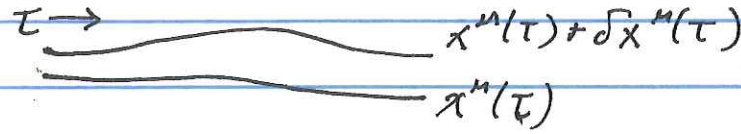
# Geodesic Deviation

Weinberg 6.10

Consider a pair of nearby geodesics in a space(time) with affine connection $\Gamma^\mu_{\nu\lambda}$.




Geodesic eqn for each path:

(1) $$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

(2) $$0 = \frac{d^2}{d\tau^2}\left[x^\mu+\delta x^\mu\right] + \Gamma^\mu_{\nu\lambda}(x+\delta x)\frac{d}{d\tau}\left[x^\nu+\delta x^\nu\right]\frac{d}{d\tau}\left[x^\lambda+\delta x^\lambda\right]$$

$$\approx \frac{d^2}{d\tau^2}\left[x^\mu+\delta x^\mu\right] + \partial_\rho\Gamma^\mu_{\nu\lambda}(x)\,\delta x^\rho\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$+\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau}$$

(2)−(1): $$\boxed{\frac{d^2\delta x^\mu}{d\tau^2} = -\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\rho - 2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{d}{d\tau}\delta x^\lambda}$$

We can write this covariantly in terms of covariant derivatives along $x^\mu(\tau)$.

$$\frac{D}{D\tau}\delta x^\mu = \frac{d}{d\tau}\delta x^\mu + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\delta x^\nu$$

$$\frac{D^2}{D\tau^2}\delta x^\mu = \frac{d}{d\tau}\left[\frac{d}{d\tau}\delta x^\mu + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\delta x^\nu\right]$$
$$+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\left(\frac{d}{d\tau}\delta x^\sigma + \Gamma^\sigma_{\nu\lambda}\frac{dx^\lambda}{d\tau}\delta x^\nu\right)$$

$$\frac{D^2}{D\tau^2}\delta x^\mu = \left(\frac{d^2\delta x^\mu}{d\tau^2} + \partial_\rho \Gamma^\mu_{\nu\lambda} \frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu + \Gamma^\mu_{\nu\lambda}\frac{d^2x^\lambda}{d\tau^2}\delta x^\nu \right.$$

$$\left. + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{d\delta x^\nu}{d\tau}\right) + \left(\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{d}{d\tau}\delta x^\sigma \right.$$

$$\left. + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu\right)$$

Using the boxed eqn. for $\frac{d^2\delta x^\mu}{d\tau^2}$ and the geodesic eqn for $\frac{d^2x^\lambda}{d\tau^2}$

$$\frac{D^2}{D\tau^2}\delta x^\mu = -\partial_\rho \Gamma^\mu_{\nu\lambda}\delta x^\rho \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \cancel{-2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau}} \cancel{+2\Gamma^\mu_{\nu\lambda}\frac{d\delta x^\nu}{d\tau}\frac{dx^\lambda}{d\tau}}$$

$$+ \partial_\rho \Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu$$

$$- \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\delta x^\nu$$

$$= \delta x^\lambda \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\underbrace{\left(\partial_\rho \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\mu_{\lambda\sigma}\Gamma^\sigma_{\rho\nu}\right)}_{R^\mu{}_{\nu\lambda\rho}}$$

$$\boxed{\frac{D^2}{D\tau^2}\delta x^\mu = \delta x^\lambda \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} R^\mu{}_{\nu\lambda\rho}}$$

Geodesic Deviation Equation

The geodesic deviation equation describes gravitational tidal forces, which are dependent on $R^\mu{}_{\nu\lambda\rho}$.

For nonrelativistic motion in a weak-field limit, $\frac{dx^\nu}{d\tau} \sim (1,0,0,0)$

$$\frac{D^2}{D\tau^2}\delta x^i \approx \boxed{\frac{d^2}{dt^2}\delta x^i \approx R^i{}_{0j0}\,\delta x^j} \approx -\frac{\partial^2\phi}{\partial x^i \partial x^j}\,\delta x^j$$

Tidal acceleration

$\phi$ = Newtonian potential.

Consider a gravitational wave with polarization tensor

$$e_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & 0 & 0 \\ 0 & 0 & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and metric $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$

with $h_{\mu\nu} = f(z-t)\, e_{\mu\nu} + \text{c.c.}$

Consider two nearby particles separated by vector $\delta x^\lambda$,

$\dfrac{dx^\nu}{d\tau} \approx \delta^\nu{}_0$. (nonrelativistic)

The geodesic deviation equation is in this limit

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu{}_{0\lambda 0}\,\delta x^\lambda$$

The relevant Christoffel symbols are $\Gamma^\mu_{0\lambda} \approx \frac{1}{2}\eta^{\mu\sigma}\partial_0 h_{\lambda\sigma} = \frac{1}{2}\partial_0 h_\lambda{}^\mu$

$\Gamma^\mu_{00} = 0$

$$R^\mu{}_{0\lambda0} = \partial_0 \Gamma^\mu_{0\lambda} - \partial_\lambda \cancel{\Gamma^\mu_{00}} + \underbrace{\Gamma^\mu_{\rho 0}\Gamma^\rho_{0\lambda} - \Gamma^\mu_{\rho\lambda}\cancel{\Gamma^\rho_{00}}}_{\mathcal{O}(h^2)}$$

$$\approx \frac{1}{2}\partial_0^2 h_\lambda{}^\mu = \frac{1}{2} f''(z-t)\, e_\lambda{}^\mu + \text{c.c.}$$

$$\frac{D^2}{D\tau^2}\delta x^1 = \frac{1}{2} f''(z-t)\, e_{11}\,\delta x^1 + \text{c.c.}$$

$$\frac{D^2}{D\tau^2}\delta x^2 = \frac{1}{2} f''(z-t)\,(-e_{11})\,\delta x^2 + \text{c.c.}$$

For nonrelativistic particles in a weak field, $\tau \approx t$, $\frac{D}{D\tau} \approx \frac{d}{dt}$.

$$\Rightarrow \frac{d^2}{dt^2}\delta x^1 = \frac{1}{2} f''(z-t)\, e_{11}\,\delta x^1 + \text{c.c.}$$

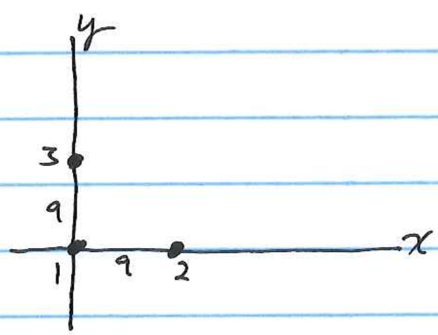
$$\frac{d^2}{dt^2}\delta x^2 = -\frac{1}{2} f''(z-t)\, e_{11}\,\delta x^2 + \text{c.c.}$$

To $\mathcal{O}(\delta x, f)^2$, the solution for small times is:

$$\boxed{\begin{aligned} \delta x^1 &= \left(1 + \left(\tfrac{1}{2} f(z-t)\,\epsilon_{11} + \text{c.c.}\right)\right) \delta x^1(0) \\ \delta x^2 &= \left(1 - \left(\tfrac{1}{2} f(z-t)\,\epsilon_{11} + \text{c.c.}\right)\right) \delta x^2(0) \end{aligned}}$$

This should be compared with our earlier analysis of gravitational waves, in which we calculated the proper distance between two nearby particles in the background of a gravitational wave.

From Before:




$$\Delta s_{12} \approx a\left(1 + \tfrac{1}{2}\left(f(z-t)\,\epsilon_+ + \text{c.c.}\right)\right)$$
$$\Delta s_{13} \approx a\left(1 - \tfrac{1}{2}\left(f(z-t)\,\epsilon_+ + \text{c.c.}\right)\right)$$

With the identification $\epsilon_{11} = \epsilon_+$, we confirm the physical interpretation of $\Delta s$ and $\delta x^i$ as physical displacements.