Physics 786, Spring 2017
Problem Set 7, Due Thursday, April 6, 2017.

1. *Newtonian Stars*

Consider static, spherically symmetric solutions to Einstein's equations for a fluid with density $\rho(r)$ and pressure $p(r)$, with metric of the form,

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

Assume the fluid is nonrelativistic, and consider the nonrelativistic limit of Einstein's equations for this system. Assume $\phi(0) = \lambda(0) = 0$, and

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right)$$

for $r \leq R$, and $\rho(r) = 0$ for $r > R$.

Find the spacetime metric for $r < R$ and $r > R$.

2. *Killing Vectors*

a) Consider 2D Euclidean space in Cartesian coordinates x, y. Find the Killing vectors related to translations and rotation about $x = y = 0$ in these coordinates.

b) What are the corresponding "constants of the motion" and their physical interpretation?

3. *Schwarzschild Spacetime*

a) A massive test particle is released from $r = R > 2GM$ in the Schwarzschild geometry (in standard coordinates), and falls radially inward. Show that the following correctly parametrizes the trajectory:

$$r = \frac{R}{2}(1 + \cos\eta)$$
$$\tau = \frac{R}{2}\left(\frac{R}{2GM}\right)^{1/2}(\eta + \sin\eta).$$

b) Show that the proper time elapsed when the particle reaches $r = 2GM$ is finite.

4. *More Schwarzschild*

A massive test particle is at $r = r_0 < 2GM$ at $t = 0$. Assume that the metric inside the horizon takes the same Schwarzschild form as outside the horizon.

a) Show that

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

b) Show that the test particle necessarily reaches $r = 0$ in finite proper time.