Physics 786, Spring 2017
Problem Set 4, Due Thursday, March 2, 2017.

1. *Index Contraction*

a) If A^μ and B^ν are vectors under general coordinate transformations, then show that $A^\mu B_\mu = A^\mu B^\nu g_{\mu\nu}$ is a scalar.

b) Show that the covariant derivative of $A^\mu B_\mu$ is

$$D_\nu(A^\mu B_\mu) = \partial_\nu(A^\mu B_\mu).$$

2. *Covariant derivative of the metric*

a) Show that $g_{\mu\nu;\lambda} = 0$.
b) Show that $g^{\mu\nu}{}_{;\lambda} = 0$.
c) Show that $\delta_\mu{}^\nu{}_{;\lambda} = 0$.

3. *Divergence in Spherical Coordinates*

Consider spherical coordinates (r, θ, ϕ), which are related to Cartesian coordinates (x, y, z) by,

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned}$$

a) If the components of a vector in Cartesian coordinates are V^x, V^y, V^z, then what are the components of that vector in coordinates (r, θ, ϕ), *i.e.* V^r, V^θ, V^ϕ?

b) Using the covariant expression for the divergence,

$$D_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}V^\mu\right),$$

calculate the divergence $\nabla\cdot\mathbf{V}$ in 3D Euclidean space in coordinates (r, θ, ϕ).

c) Your result in part (b) might not look like the expression for the divergence in spherical coordinates that you are familiar with. What is the relation between V^r, V^θ, V^ϕ and the components of the vector $\mathcal{V}^r$, $\mathcal{V}^\theta$ and $\mathcal{V}^\phi$ in terms of an orthonormal basis of unit vectors in spherical coordinates, *i.e.* $\mathbf{V} = \mathcal{V}^r\,\hat{\mathbf{e}}_r + \mathcal{V}^\theta\,\hat{\mathbf{e}}_\theta + \mathcal{V}^\phi\,\hat{\mathbf{e}}_\phi = V^x\,\hat{\mathbf{e}}_x + V^y\,\hat{\mathbf{e}}_y + V^z\,\hat{\mathbf{e}}_z$? Now explain how the result of part (b) agrees with the usual expression for $\nabla \cdot \mathbf{V}$ in spherical coordinates.