Physics 786, Spring 2017
Problem Set 3 Due Thursday, February 23, 2017.

1. *Electromagnetic Waves*

We have seen several analogies between electromagnetism and general relativity. Here we will consider electromagnetic waves in a relativistic framework.

In terms of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, half of the source-free Maxwell's equations take the form

$$\partial_\mu F^{\mu\nu} = 0.$$

a) Show that $F_{\mu\nu}$ is invariant under gauge transformations $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu f(x)$ for any function $f(x)$.

b) Show that a gauge transformation can generically be chosen to enforce the Lorenz gauge condition, $\partial_\mu A^\mu = 0$. What is the residual gauge freedom that remains after the Lorenz gauge condition is imposed?

c) In Lorenz gauge, Maxwell's equations reduce to the wave equation for each component of A^μ. Assume a plane-wave solution of the form

$$A^\mu(x) = \epsilon^\mu \, e^{ik\cdot x} + \epsilon^{\mu\,*} \, e^{-ik\cdot x}.$$

What conditions on k^μ and ϵ^μ follow from the wave equation and the Lorenz gauge condition?

d) With the choice $k^\mu = (k, 0, 0, k)^\mu$, show that the polarization vectors $\epsilon^\mu_{(1)} = (0, 1, 0, 0)^\mu$, $\epsilon^\mu_{(2)} = (0, 0, 1, 0)^\mu$ and $\epsilon^\mu_{(3)} = (1, 0, 0, 1)^\mu$ satisfy the Lorenz gauge condition. Explain why the solution with $\epsilon_{(3)}$ is unphysical.

e) By considering a rotation about the direction of motion, find the helicities of the plane waves with polarization vectors $\epsilon^\mu_\pm \equiv \epsilon^\mu_{(1)} \pm i\epsilon^\mu_{(2)}$ and $\epsilon_{(3)}$. What are the helicities of the physical solutions?

2. *Spherical coordinates*

Consider the coordinates $x = r \sin\theta \, \cos\phi$, $y = r \sin\theta \, \sin\phi$, $z = r \cos\theta$ in Euclidean space with length element

$$ds^2 = dx^2 + dy^2 + dz^2.$$

a) Calculate the length element, and hence the components of the metric, in spherical coordinates.

b) Calculate the determinant of the metric, and compare with the Jacobian of the transformation $(x, y, z) \to (r, \theta, \phi)$.

c) Write expressions for $\nabla \cdot \mathbf{V}$, $\nabla \times \mathbf{V}$, $\nabla\phi$ and $\nabla^2\phi$ for vector field $\mathbf{V}$ and scalar field ϕ, in spherical coordinates, and remind yourself how you have derived these formulae in the past. What are the basis unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$?