Weinberg 4.8

Coordinate Transformations and Orthogonal Coordinates

In Euclidean space it is common to consider non-Cartesian coordinate systems, often to take advantage of a certain symmetry such as rotational invariance. Examples are polar coordinates and spherical coordinates.

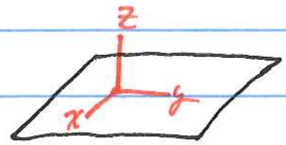
In an orthogonal coordinate system the metric is diagonal:

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = h_i^2 \delta_{ij} \quad \text{(not summed over } i\text{)}$$

functions of coordinates.

The basis vectors in this case are orthogonal, but are not necessarily unit vectors.

Example: Consider 2D Euclidean space, described as the x-y plane embedded in 3D Euclidean space.



Points on the plane are described by

$$\vec{X}(x,y) = x\,\hat{e}_x + y\,\hat{e}_y + 0\,\hat{e}_z$$

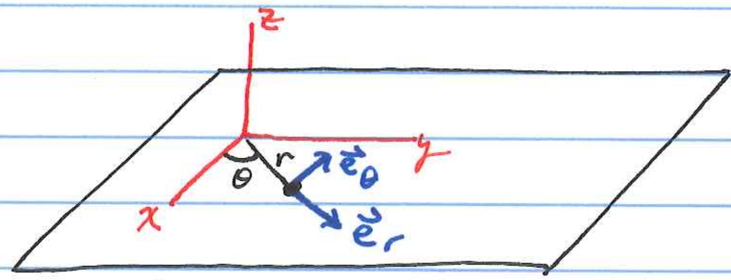
unit vector in 3D Euclidean space.

Basis vectors in Cartesian coordinates:

$$\left.\begin{aligned} \vec{e}_x &= \frac{\partial \vec{X}}{\partial x} = \hat{e}_x \\ \vec{e}_y &= \frac{\partial \vec{X}}{\partial y} = \hat{e}_y \end{aligned}\right\} \quad \begin{aligned} g_{ij} &= \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \\ ds^2 &= dx^2 + dy^2 \end{aligned}$$

Now consider polar coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \cos\theta &= x/\sqrt{x^2+y^2} \end{aligned}$$



Polar coordinates: $\vec{X}(r,\theta) = r\cos\theta\,\hat{e}_x + r\sin\theta\,\hat{e}_y + 0\,\hat{e}_z$

Basis vectors: $\vec{e}_r = \dfrac{\partial\vec{X}}{\partial r} = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y + 0\,\hat{e}_z$

$$\vec{e}_\theta = \frac{\partial\vec{X}}{\partial\theta} = -r\sin\theta\,\hat{e}_x + r\cos\theta\,\hat{e}_y + 0\,\hat{e}_z$$

$g_{ij} = \vec{e}_i\cdot\vec{e}_j = h_i^2\,\delta_{ij}$, where

$$\left.\begin{array}{l} h_r^2 = \vec{e}_r\cdot\vec{e}_r = 1 \\ h_\theta^2 = \vec{e}_\theta\cdot\vec{e}_\theta = r^2 \end{array}\right\} ds^2 = dr^2 + r^2 d\theta^2$$

An orthonormal basis of vectors is obtained by dividing by the appropriate factor:

$$\hat{e}_i = \frac{\vec{e}_i}{|\vec{e}_i|} = \frac{\vec{e}_i}{h_i} = \frac{\vec{e}_i}{\sqrt{g_{ii}}}$$

In polar coordinates: $\hat{e}_r = \vec{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y$

$$\hat{e}_\theta = \frac{\vec{e}_\theta}{r} = -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y$$

Under coordinate transformations the basis vectors are covariant vectors:

$$\vec{e}_{i'} = \frac{\partial\vec{X}}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^{i'}}\frac{\partial\vec{X}}{\partial x^j} = \frac{\partial x^j}{\partial x^{i'}}\,\vec{e}_j$$

If V^i is a contravariant vector, then $V^i\vec{e}_i$ is a scalar, i.e. coordinate-invariant

The vector $\vec{V} = V^x \vec{e}_x + V^y \vec{e}_y$ becomes in (r,θ)-coordinates
$\vec{V} = V^r \vec{e}_r + V^\theta \vec{e}_\theta$, where

$$V^r = \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y = \cos\theta \, V^x + \sin\theta \, V^y$$
$$V^\theta = \frac{\partial \theta}{\partial x} V^x + \frac{\partial \theta}{\partial y} V^y = -\frac{1}{r} \sin\theta \, V^x + \frac{1}{r} \cos\theta \, V^y$$
(Exercise)

With $\vec{e}_r = \cos\theta \, \vec{e}_x + \sin\theta \, \vec{e}_y$ and
$\vec{e}_\theta = -r \sin\theta \, \vec{e}_x + r \cos\theta \, \vec{e}_y$ as calculated earlier,
it is straightforward to check that $V^r \vec{e}_r + V^\theta \vec{e}_\theta = V^x \vec{e}_x + V^y \vec{e}_y$.

In terms of unit basis vectors, we would write
$$\begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{V}^r \\ \mathbb{V}^\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V^x \\ V^y \end{pmatrix}$$

so that $V^x \hat{e}_x + V^y \hat{e}_y = \mathbb{V}^r \hat{e}_r + \mathbb{V}^\theta \hat{e}_\theta$
$$= \mathbb{V}^r \frac{\vec{e}_r}{\sqrt{g_{rr}}} + \mathbb{V}^\theta \frac{\vec{e}_\theta}{\sqrt{g_{\theta\theta}}}.$$

Hence, the components of $\vec{V}$ in <u>orthonormal</u> polar coordinates are related to the components of $\vec{V}$ in (r,θ)-coordinates obtained by a coordinate transformation from (x,y)-coordinates by
$$\mathbb{V}^r = \sqrt{g_{rr}} \, V^r, \quad \mathbb{V}^\theta = V^\theta \sqrt{g_{\theta\theta}}.$$

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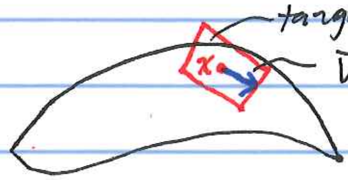
Covariant Derivative

Vector fields on a manifold live in the tangent space, which in the case that the manifold is embedded in a higher-dimensional Euclidean space we can think of as the tangent plane at each point in the manifold.



we use Greek indices, but the discussion is essentially the same in space or spacetime.

Ordinary derivatives of $\vec{w}(x)$ do not generally remain in the tangent space:

$$\partial_\nu \vec{W}(x) = \partial_\nu(W^\mu \vec{e}_\mu) = (\partial_\nu W^\mu)\,\vec{e}_\mu + W^\mu \partial_\nu \vec{e}_\mu$$

or, using $\partial_\nu \vec{e}_\mu = \Gamma^\lambda_{\mu\nu}\,\vec{e}_\lambda + K_{\mu\nu}\,\hat{n}$,

$$\partial_\nu \vec{w} = (\partial_\nu W^\mu)\vec{e}_\mu + W^\lambda \Gamma^\mu_{\lambda\nu}\,\vec{e}_\mu + W^\mu K_{\mu\nu}\,\hat{n}$$
$$= (\partial_\nu W^\mu + \Gamma^\mu_{\lambda\nu} W^\lambda)\,\vec{e}_\mu + W^\mu K_{\mu\nu}\,\hat{n}$$

The projection of $\partial_\nu \vec{W}$ onto the tangent plane defines the covariant derivative of $\vec{w}(x)$:

$$\boxed{D_\nu \vec{W} \equiv (\partial_\nu W^\mu + \Gamma^\mu_{\lambda\nu} W^\lambda)\,\vec{e}_\mu \equiv (D_\nu W^\mu)\,\vec{e}_\mu}$$

Note that $D_\nu W^\mu$ depends in general on all components of W^α, not just the component $\alpha = \mu$.

The covariant derivative $D_\nu \vec{W}$ lives in the tangent space, and $D_\nu W^\mu$ transforms as a tensor. This is another way to introduce the covariant derivative: We ask for a derivative that transforms covariantly under coordinate transformations.

The basic point is that neither $\partial_\mu W^\alpha$ nor $\Gamma^\mu_{\nu\lambda}$ is a tensor, but the combination $D_\nu W^\mu$ is.

The Affine Connection is not a tensor under general coordinate transformations.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$, $\xi^{\alpha}(x)$ is a locally inertial coordinate system.

In the coordinate system x',

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

chain rule

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right)$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right]$$

$$\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

$$\boxed{\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}}$$

Differentiation of a tensor does not generally yield another tensor.

Under the transformation $x \to x'$, $V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$ for vector V^{μ}.

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}}}_{\text{Tensor-like transformation}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}}_{\text{non-tensorlike.}}$$

semicolon, not the letter ;

However, the combination $D_\lambda V^\mu \equiv V^\mu{}_{;\lambda} = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} V^\kappa$,

the covariant derivative, is a tensor:

$$V'^\mu{}_{;\lambda} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial x'^\lambda} V^\nu{}_{;\rho}$$

To show this we rewrite the transformation of $\Gamma^\lambda_{\mu\nu}$ in a different way.

Use $\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu} = \delta^\lambda_\nu$

$$\frac{\partial}{\partial x'^\mu} : \quad \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma} = 0$$

$$\Rightarrow \boxed{\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}}$$

It is now straightforward to show that the non-tensorlike term in the transformation of $\Gamma^\lambda_{\mu\nu}$ cancels the non-tensorlike term in the transformation of $\partial_\lambda V^\mu$ in the combination $\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\kappa} V^\kappa$. (Exercise)

Similarly, the covariant derivative of a covariant vector is

$$\boxed{D_\nu V_\mu \equiv V_{\mu;\nu} \equiv \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda}$$

Under a coordinate transformation $D_\nu V_\mu$ transforms as a tensor:

$$V'_{\mu;\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} V_{\rho;\sigma} \qquad \text{(Exercise)}$$

In general, covariant derivatives of tensors involve a sum of terms, each involving one factor of $\Gamma^{\lambda}_{\mu\nu}$, one for each index on the tensor.

Example: $T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\sigma}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}{}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}{}_{\lambda} - \Gamma^{\kappa}_{\lambda\rho} T^{\mu\sigma}{}_{\kappa}$

Exercise: Check that $T^{\mu\sigma}{}_{\lambda;\rho}$ is a tensor.

Properties of Covariant Derivatives

1) $(\alpha A^{\mu}{}_{\nu} + \beta B^{\mu}{}_{\nu})_{;\lambda} = \alpha A^{\mu}{}_{\nu;\lambda} + \beta B^{\mu}{}_{\nu;\lambda}$ linearity

2) $(A^{\mu}{}_{\nu} B^{\lambda})_{;\rho} = A^{\mu}{}_{\nu;\rho} B^{\lambda} + A^{\mu}{}_{\nu} B^{\lambda}{}_{;\rho}$ Leibniz rule

3) $T^{\mu\lambda}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\lambda}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\lambda}{}_{\lambda}$ Derivative of contraction works as if contracted indices were not there

Covariant Differentiation of the Metric

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}$$

$$= 0 \quad \text{(using definition of } \Gamma^{\rho}_{\lambda\mu} \text{ in terms of } g_{\mu\nu}\text{)}$$

We can also show this by considering a locally inertial coordinate system, in which $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = 0$, $\Gamma^{\rho}_{\lambda\mu} = 0$ at some pt P. But $g_{\mu\nu;\lambda}$ is a tensor, so in a general coordinate system, $g_{\mu\nu;\lambda}$ remains zero.

Similarly, $g^{\mu\nu}{}_{;\lambda} = 0$

$\delta^{\mu}{}_{\nu;\lambda} = 0$

* Covariant Differentiation Commutes w/ Raising, Lowering Indices

$$(g^{\mu\nu} V_\nu)_{;\lambda} = \cancel{g^{\mu\nu}{}_{;\lambda}} V_\nu + g^{\mu\nu} V_{\nu;\lambda}$$

$$= g^{\mu\nu} V_{\nu;\lambda}$$

Special Cases of Covariant Differentiation

Covariant Derivative of a Scalar S: $\boxed{S_{;\mu} = \frac{\partial S}{\partial x^\mu}}$

Covariant Curl: Recall $V_{\mu;\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda$

symmetric in $\mu \leftrightarrow \nu$.

curl: $\boxed{V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}}$ = ordinary curl.

The covariant divergence of a covariant vector can be written in terms of $g = \det(g_{\mu\nu})$ using the following identity:

$$\boxed{\mathrm{Tr}\left\{M^{-1}(x) \frac{\partial}{\partial x^\lambda} M(x)\right\} = \frac{\partial}{\partial x^\lambda} \ln \det M(x)}$$

Proof: If $x^\lambda \to x^\lambda + \delta x^\lambda$, then

$$\delta \ln \det M = \ln \det(M + \delta M) - \ln \det M$$

$$\delta \ln \det M = \ln\left(\frac{\det(M+\delta M)}{\det M}\right)$$

$$= \ln \det\left(M^{-1}(M+\delta M)\right)$$

$$= \ln \det\left(\mathbb{1} + M^{-1}\delta M\right)$$

$$\approx \ln\left(1 + \operatorname{Tr} M^{-1}\delta M\right)$$

$$\approx \operatorname{Tr} M^{-1}\delta M$$

$$\lim_{\delta x^\lambda \to 0} \frac{\delta \ln \det M}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} \ln \det M$$

$$= \operatorname{Tr}\left(M^{-1}\frac{\partial M}{\partial x^\lambda}\right) \quad \blacksquare$$

with $M = g_{\rho\mu}$, $g^{\mu\rho}\frac{\partial}{\partial x^\lambda} g_{\rho\mu} = \frac{\partial}{\partial x^\lambda} \ln g$

$$= \frac{2}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}\sqrt{g}$$

Then $\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\left\{\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\mu\lambda}\right\}$

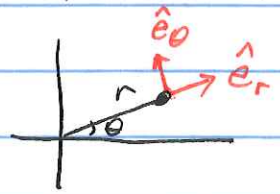
$$= \frac{1}{2} g^{\mu\rho}\,\partial_\lambda g_{\rho\mu}$$

$$\boxed{\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{g}}\partial_\lambda \sqrt{g}}$$

$$\Longrightarrow \boxed{V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}\,V^\mu}$$

Example: Divergence in 2D Polar Coordinates

$\partial_\mu V^\mu$ is not a scalar under general coordinate transformations. $D_\mu V^\mu$ is a scalar — it takes the same value in any coordinate system.



Polar Coordinates:

$x = r\cos\theta \quad \Leftrightarrow \quad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \quad\quad \cos\theta = x/\sqrt{x^2+y^2}$

$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$. Let (x,y) be the unprimed coords, (r,θ) the primed coords.

$$V^r = \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y = \cos\theta\, V^x + \sin\theta\, V^y$$

$$V^\theta = \frac{\partial \theta}{\partial x} V^x + \frac{\partial \theta}{\partial y} V^y = -\frac{\sin\theta}{r} V^x + \frac{\cos\theta}{r} V^y$$

$$D_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, V^\mu\right) = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} = \nabla\cdot\vec{V}$$

$$ds^2 = dr^2 + r^2 d\theta^2 \rightarrow g = r^2 \quad (= \det g_{\mu\nu})$$

$$D_\mu V^\mu = \frac{1}{r}\left[\frac{\partial}{\partial r}(r V^r) + r \frac{\partial}{\partial \theta} V^\theta\right]$$

In terms of orthonormal basis vectors $\hat{e}_r$, $\hat{e}_\theta$, the vector

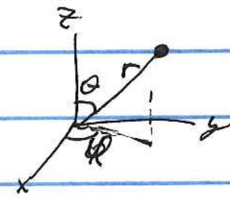
$$\vec{V} = \underbrace{\sqrt{g_{rr}}\, V^r}_{\mathbb{V}^r = V^r} \hat{e}_r + \underbrace{\sqrt{g_{\theta\theta}}\, V^\theta}_{\mathbb{V}^\theta = r V^\theta} \hat{e}_\theta$$

We recover the usual expression for the divergence in polar coordinates: $\boxed{\nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\mathbb{V}^r) + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbb{V}^\theta}$

Covariant Laplacian / D'Alembertian

If $\phi(x)$ is a scalar, $\phi_{;\mu}{}^{;\mu} = \left(g^{\mu\nu}\phi_{;\nu}\right)_{;\mu}$
$$= \left(g^{\mu\nu}\partial_\nu \phi\right)_{;\mu}$$
$$\boxed{\phi_{;\mu}{}^{;\mu} = \frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu \phi\right)}$$

In flat space these formulae allow us to compute the gradient, divergence, and curl in arbitrary coordinates.



Example: Laplacian in spherical coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/(r^2\sin^2\theta) \end{pmatrix}$$

$$g \equiv \det g_{\mu\nu} = r^4 \sin^2\theta$$

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\,\partial_i\left(\sqrt{g}\, g^{ij}\,\partial_j \phi\right)$$

$$= \frac{1}{r^2\sin\theta}\left\{ \frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(r^2\sin\theta\cdot\frac{1}{r^2}\,\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(r^2\sin\theta\cdot\frac{1}{r^2\sin^2\theta}\,\frac{\partial\phi}{\partial\varphi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$