

## Einstein's Geometric Interpretation of Gravitation

The field-theoretic approach to a theory of gravitation has become unwieldy, so we now turn to Einstein's theory, which is equivalent order-by-order in how to the field theory we have been describing until now.

In the geometric approach, we begin by conjecturing that all particles which are free except for the influence of gravitation fall along trajectories that stationarize the proper time,  $d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$ .

The paths which stationarize the distance between points on a curved space, e.g. the great circles on the 2-sphere, are called geodesics. By analogy, curves which stationarize the proper time are called geodesics in spacetime. Hence, particles fall along geodesics in the spacetime described by the metric  $g_{\mu\nu}$ . As we have seen, these geodesics satisfy the geodesic equation

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0}, \text{ where } \tau \text{ parametrizes the trajectory.}$$

(For lightlike trajectories  $d\tau^2 = 0$  along the trajectory, so we instead parametrize the trajectory by some other parameter  $\sigma$  or  $\xi^0$  and we write

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = 0.$$

↑  
time coordinate  
in freely-falling frame

In different coordinate systems  $g_{\mu\nu}$  looks different. If  $dt^2 = dt^2 - d\vec{x}^2$ , so that  $g_{\mu\nu} = \eta_{\mu\nu}$ , then in spherical coordinates  $dt^2 = dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$ , so  $g_{00} = 1, g_{rr} = 1, g_{\theta\theta} = r^2, g_{\phi\phi} = r^2 \sin^2 \theta$ .

These different forms of  $g_{\mu\nu}$  describe the same spacetime, which we call Minkowski spacetime, or "Minkowski space" for short.

In general relativity, the existence of gravity is tantamount to the inability to transform  $g_{\mu\nu}$  to  $\eta_{\mu\nu}$  by a coordinate transformation. We need to find a way to test whether or not  $g_{\mu\nu}$  can be transformed to  $\eta_{\mu\nu}$ , i.e. whether or not there is gravitation.

By Einstein's equivalence principle, it should be possible to choose coordinates  $x^{\mu}$  such that in the neighborhood of a spacetime point  $P$  the spacetime is locally equivalent to Minkowski space. More precisely, we will see that in general we can choose coordinates such that at a pt.  $P$ ,  $g_{\mu\nu}(x_P) = \eta_{\mu\nu}$ , and  $\Gamma^{\mu}_{\alpha\beta}(x_P) = 0$ . These are the locally inertial, or freely falling, coordinates at point  $P$ .

As a warmup for the geometric concepts that will be relevant for Einstein's description of gravity, we will consider some familiar spaces.

Consider a 2-sphere, embedded in 3-dimensional Euclidean space.



$$x^2 + y^2 + z^2 = R^2 \rightarrow z = \sqrt{R^2 - x^2 - y^2}$$

$ds^2 = dx^2 + dy^2 + dz^2$  metric of Euclidean space in Cartesian coordinates.

Describe points on the sphere by coordinates  $x, y$ . Note that a given  $(x, y)$  corresponds to two points on the sphere, related by  $z \rightarrow -z$ . Hence these coordinates should only be used to describe  $z \geq 0$  or  $z \leq 0$  on the sphere.

# It will generally be necessary to patch together different coordinate systems, each of which covers only part of the geometry.

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= - \frac{(x dx + y dy)}{\sqrt{R^2 - x^2 - y^2}} \end{aligned}$$

$$(dz)^2 = \frac{x^2 dx^2 + y^2 dy^2 + 2xy dx dy}{R^2 - x^2 - y^2}$$

Hence, on the 2-sphere,

$$dS^2 = dx^2 + dy^2 + dz(x,y)^2$$
$$= \left(1 + \frac{x^2}{R^2 - x^2 - y^2}\right) dx^2 + \left(1 + \frac{y^2}{R^2 - x^2 - y^2}\right) dy^2 + \left(\frac{2xy}{R^2 - x^2 - y^2}\right) dx dy$$

$$\equiv g_{ij} dx^i dx^j, \text{ where } x^1 = x, x^2 = y.$$

In other words,

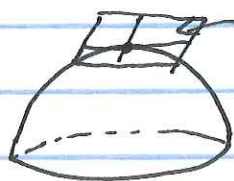
$$g_{xx} = 1 + \frac{x^2}{R^2 - x^2 - y^2}, \quad g_{yy} = 1 + \frac{y^2}{R^2 - x^2 - y^2},$$

$$g_{xy} = g_{yx} = \frac{xy}{R^2 - x^2 - y^2}$$

- valid for  $x^2 + y^2 < R^2$ , i.e.  $z > 0$  or  $z < 0$ .

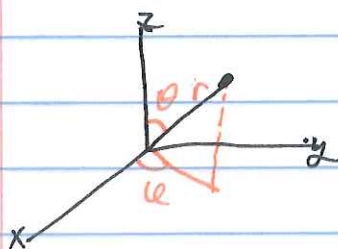
At  $z=1$ ,  $x=y=0$ ,

$$\left. \begin{aligned} g_{xx} = g_{yy} = 1, \quad g_{xy} = 0 = g_{yx} \\ \frac{\partial g_{ij}}{\partial x^k} \Big|_{x=y=0} = 0 \end{aligned} \right\} \text{locally flat coordinates} \\ \text{at point } (x,y) = (0,0).$$



Cartesian coordinates  $x, y$  on plane tangent to sphere at  $(x,y) = (0,0)$ .

## Spherical Coordinates



$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (\text{Exercise})$$

2-sphere:  $r = R$

$$\Rightarrow ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$g_{\theta\theta} = R^2, \quad g^{\theta\theta} = R^{-2}$$

$$g_{\varphi\varphi} = R^2 \sin^2 \theta, \quad g^{\varphi\varphi} = \frac{1}{R^2 \sin^2 \theta}$$

(Note that  $g^{\theta\theta} = \frac{1}{g_{\theta\theta}}$  and  $g^{\varphi\varphi} = \frac{1}{g_{\varphi\varphi}}$  only because the metric is diagonal, i.e.  $g_{\theta\varphi} = g_{\varphi\theta} = 0$ .)

The nonvanishing Christoffel symbols are

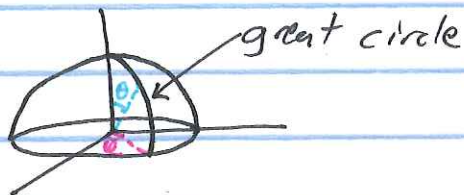
$$\Gamma_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin 2\theta \quad (\text{Exercise})$$

$$\Gamma_{\theta\theta}^{\varphi} = \cot \theta$$

$$\Rightarrow \left. \begin{aligned} \frac{d^2\varphi}{ds^2} + (\cot \theta) \frac{d\varphi}{ds} \frac{d\theta}{ds} &= 0 \\ \frac{d^2\theta}{ds^2} + \left(-\frac{1}{2} \sin 2\theta\right) \left(\frac{d\varphi}{ds}\right)^2 &= 0 \end{aligned} \right\} \text{Geodesic equations}$$

The solutions are the great circles, for example the great circles through the north pole,

$$\begin{cases} \varphi = \text{constant} & \rightarrow \frac{d\varphi}{ds} = 0 = \frac{d^2\varphi}{ds^2} \\ \theta = \alpha s & \rightarrow \frac{d^2\theta}{ds^2} = 0 \end{cases}$$



If we lived on this 2-sphere, what evidence would we have of the non-flatness of space?

1) The distance between nearby geodesics would not be constant  $\rightarrow$  there are no "parallel" lines.



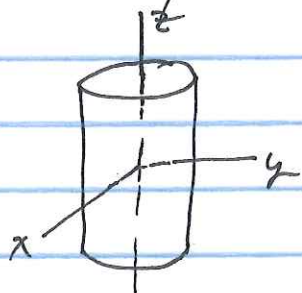
2) If we drew a circle (locus of points

⊖ equidistant from some point) and measured the radius  $R$ , the circumference  $C$  would satisfy  $C < 2\pi R$ .

3) If we drew a triangle formed by 3 geodesics, the sum of the internal angles would not sum to  $180^\circ$ . Instead, the sum of the angles would grow with the area in the triangle.



For comparison, consider a cylinder,



$$x^2 + y^2 = R^2$$

In polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$   
 $r = R$  on the cylinder.

$$ds^2 = dz^2 + R^2 d\theta^2 \quad (\text{Exercise})$$

Rescaling the angular coordinate  $\varphi \equiv R\theta$ ,

$$ds^2 = dz^2 + d\varphi^2$$

This is the Euclidean metric  $\rightarrow$  the cylinder is flat!  
Note that  $\theta \in [0, 2\pi)$ , so the cylinder is not globally the same as Euclidean space, but around any point it is.

This is easily understood. Imagine cutting the cylinder along a vertical seam (as drawn above). Unwrapping the cylinder, it should be clear that the cylinder is like a flat sheet of paper, just glued along the seam.

If we think of the cylinder as embedded in 3 dimensions, it appears to have a curvature. This curvature is referred to as extrinsic to the cylinder.

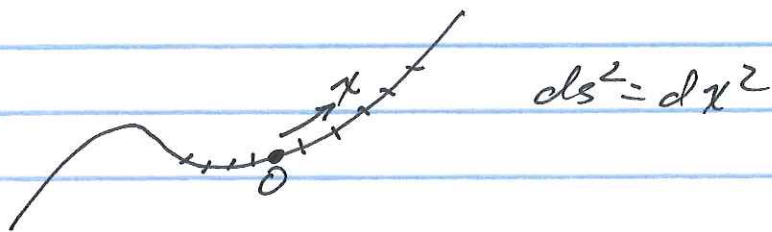
If we consider just the cylinder without reference to its embedding in 3 dimensions, the cylinder is flat, i.e. it has no intrinsic curvature.

We can perform the same experiments on the cylinder to try to detect the effects of curvature as we imagined on the sphere:

- 1) The distance between geodesics can be constant.  
(These are parallel lines)
- 2) Circles satisfy  $C = 2\pi R$ .
- 3) The sum of the angles in a triangle is  $180^\circ$ .

To see these facts, it suffices to consider the situation when the cylinder is unwrapped.

Similarly, any 1-dimensional curve has vanishing intrinsic curvature. After all, we can parametrize points on a curve by their distance along the curve from some point, the origin of our coordinate system.





see 1.6

## Locally Flat Coordinates

In the examples of the 2-sphere and the cylinder, we were able to find coordinates such that at a given point  $P$ ,  $g_{ij} = \delta_{ij}$  and  $\Gamma_{jk}^i = 0$ . Such coordinates are called locally flat at point  $P$ .

Locally flat coordinates can be constructed at any point  $P$  in a sufficiently smooth space. The argument is as follows:

Call the point  $P$  the origin,  $x=0$ . Expand the metric about  $x=0$ :

$$g_{ij}(x) = g_{ij}(0) + A_{ijk} x^k + B_{ijkl} x^k x^l + \dots,$$

where  $A_{ijk}$ ,  $B_{ijkl}$ ,  $\dots$ , are expansion coefficients.

Change coordinates to  $x'^m$ , where

$$x^i = K^i_j x'^j + L^i_{jk} x'^j x'^k + \dots$$

$$\left. \frac{\partial x^i}{\partial x'^j} \right|_{x=0} = K^i_j, \text{ so } g'_{kl}(0) = g_{ij}(0) K^i_k K^j_l.$$

As a matrix equation,  $g' = K^T g K$ . The matrix  $g$  is symmetric and real  $\rightarrow$  can find  $K$  to diagonalize  $g$ .

Once  $g'_{ij}(0)$  is diagonal, rescale the coordinates such that  $g'_{ij} = \delta_{ij}$ .

Count! In  $D$  dimensions  $K$  is a  $D \times D$  matrix, has  $D^2$  elements  
 $g$  is  $D \times D$  real, symmetric, fixed to  $\mathbb{1}$  at pt.  $P \rightarrow \frac{D(D+1)}{2}$  conditions.

Hence, there are  $D^2 - \frac{D(D+1)}{2} = \frac{D(D-1)}{2}$  unused degrees of freedom in  $K$ .

This corresponds to the freedom to rotate the coordinates in any of the  $\frac{D(D-1)}{2}$  planes in  $D$  dimensions while leaving  $g_{ij}(0) = \delta_{ij}$  fixed.

At this point  $g'_{ij}(x) = \delta_{ij} + \tilde{A}'_{ijk} x^k + \tilde{B}'_{ijkl} x^l x^k + \dots$ , where the tildes indicate that after the coordinate change the expansion coefficients are in general different.

Now we can use the  $L^i{}_{jk}$  (symmetric in  $j, k$ ) in  $x^i = K^i{}_{j'} x'^{j'} + L^i{}_{jk} x'^j x'^k + \dots$

to eliminate the  $\tilde{A}'_{ijk}$  (symmetric in  $i, j$ ).

Both  $L^i{}_{jk}$  and  $\tilde{A}'_{ijk}$  have  $\frac{D(D+1)}{2} \cdot D = \frac{D^2(D+1)}{2}$  independent components.

Hence, the metric near pt.  $P$  can be chosen to take the form  $\boxed{g_{ij}(x) = \delta_{ij} + \mathcal{O}(x^2)}$

These are the locally flat coordinates.

The generalization to spacetime is that at any point  $P$  the metric can be chosen to take the form

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2)}$$

These are locally inertial coordinates at  $P$ .

## General Coordinate Transformations

Suppose we change coordinates by an arbitrary transformation  $x^M \rightarrow x'^M(\{x^\alpha\})$ .

$$dx^M \rightarrow \boxed{dx'^M = \frac{\partial x'^M}{\partial x^\alpha} dx^\alpha}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} dx'^\alpha dx'^\beta \quad (\text{chain rule})$$

$$= g'_{\alpha\beta}(x') dx'^\alpha dx'^\beta, \text{ where}$$

$$\boxed{g'_{\alpha\beta} \equiv \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}}$$

For example, for a Lorentz transformation  $x'^M = \Lambda^M{}_\nu x^\nu$ ,  
 $x^\mu = (\Lambda^{-1})^\mu{}_\alpha x'^\alpha$

$$\frac{\partial x^\mu}{\partial x'^\alpha} = (\Lambda^{-1})^\mu{}_\alpha, \quad g'_{\alpha\beta} = (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta g_{\mu\nu}.$$

This is how we expect a  $(0,2)$ -tensor to transform.

We now define tensors by their transformation properties under general coordinate transformations.

Example:  $dx'^M = \frac{\partial x'^M}{\partial x^\alpha} dx^\alpha$  contravariant vector

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \quad \text{covariant tensor}$$

(Pay close attention to where the primes go.)