Feynman 5.5, 6.1

Conservation of $T_{\mu\nu}$?

Conservation of $T_{\mu\nu}$ played an important role in our determination of the equation of motion for the gravitational field $h_{\mu\nu}$. A consequence of $\partial_\mu T^{\mu\nu} = 0$ is time-independence of the energy of the system described by $T^{\mu\nu}$. However, gravitational radiation carries off energy (it can do work), so either the total energy of gravity + matter is not conserved, or the matter energy alone is not conserved. To examine this issue we will consider the $T_{\mu\nu}$ of a gravitating particle.

To determine $T_{\mu\nu}$, we consider the relativistic relations for the energy and momentum of a particle:

Energy $\quad E = mc^2\gamma = mc^2 \frac{dt}{d\tau}$, where $d\tau^2 = dt^2 - \frac{1}{c^2} d\vec{x}^2$

$$E = \int d^3x \; \underbrace{mc^2 \frac{dt}{d\tau} \delta^3(\vec{x} - \vec{x}(t))}_{c^2 T^{00} = \text{Energy density}}$$

Momentum $\quad p^i = m \frac{dx^i}{dt}\gamma = m \frac{dx^i}{dt}\frac{dt}{d\tau} = m\frac{dx^i}{d\tau}$

$$= \int d^3x \; \underbrace{m \frac{dx^i}{d\tau} \delta^3(\vec{x} - \vec{x}(t))}_{T^{0i}}$$

$$T^{00} = m \frac{dt}{d\tau} \delta^3(\vec{x} - \vec{x}(t)) = m \frac{dt}{d\tau}\frac{dt}{d\tau}\frac{d\tau}{dt} \delta^3(\vec{x} - \vec{x}(t))$$

$$T^{0i} = m \frac{dx^i}{d\tau} \delta^3(\vec{x} - \vec{x}(t)) = m \frac{dt}{d\tau}\frac{dx^i}{d\tau}\frac{d\tau}{dt} \delta^3(\vec{x} - \vec{x}(t))$$

A Lorentz-covariant form of $T^{\mu\nu}$ consistent with these T^{00} and T^{0i} is

$$\boxed{T^{\mu\nu} = m \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta^3(\vec{x} - \vec{x}(t)) \frac{d\tau}{dt}.}$$

This is the energy-momentum tensor, or stress-energy tensor, of a particle of mass m moving along a trajectory $x^\mu(\tau)$. To make the covariance of $T^{\mu\nu}$ explicit we can write

$$T^{\mu\nu} = m \int dt\, \delta^4(x - x(t)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \frac{d\tau}{dt}$$

$$\boxed{T^{\mu\nu} = m \int d\tau\, \delta^4(x - x(\tau)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

Now consider conservation of this $T^{\mu\nu}$:

$$\partial_\mu T^{\mu\nu} = m \int d\tau\, \underbrace{\partial_\mu \delta^4(x - x(\tau)) \frac{dx^\mu}{d\tau}}_{\frac{d}{d\tau}\delta^4(x-x(\tau))} \frac{dx^\nu}{d\tau}$$

$$\overset{\text{by parts}}{=} -m \int d\tau\, \delta^4(x - x(\tau)) \frac{d^2 x^\nu}{d\tau^2}$$

$$= -m \int d\tau\, \delta^4(x - x(\tau)) \left(-\Gamma^\nu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right)$$

by the eq. of motion for the particle (the geodesic eq.)

$$= +\Gamma^\nu_{\alpha\beta} T^{\alpha\beta}$$

$$\Rightarrow \quad \boxed{\partial_\mu T^{\mu\nu} = +\Gamma^\nu_{\alpha\beta} T^{\alpha\beta}}$$

In our linear theory, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$$

so that $g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}{}_{\alpha} + O(h^2)$. (Recall that $g^{\mu\nu}$ with upper indices is defined as the inverse (as a matrix) of $g_{\mu\nu}$ with lower indices.)

$$(*) \quad \partial_\mu T^{\mu\nu} = + \Gamma^{\nu}_{\alpha\beta} T^{\alpha\beta}$$

$$= + \frac{1}{2} \eta^{\nu\rho} (\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}) T^{\alpha\beta} + O(h^2)$$

$$\neq 0$$

Evidently, $T^{\mu\nu}$ of matter alone is not conserved, but we can attempt to find a gravitational contribution to the stress-energy tensor such that the sum of the matter + gravity contributions is conserved, at least to lowest order in $h_{\mu\nu}$.

We assume that the equation $(*)$ is valid more generally, for example for a collection of particles.

We also assume that to lowest order in $h_{\mu\nu}$, the linearized Einstein equations are satisfied. This allows us to replace $T^{\alpha\beta}$ on the right-hand side of $(*)$ by derivatives of $h_{\mu\nu}$.

We can then find tensors $\chi^{\mu\nu}$ bilinear in h and its derivatives such that

$$\partial_\mu (T^{\mu\nu} + \chi^{\mu\nu}) = 0, \text{ i.e.}$$

$$\boxed{\partial_\mu \chi^{\mu\nu} = -\frac{1}{2} \eta^{\nu\rho} (\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}) T^{\alpha\beta} + O(h^2)}$$

The most general expression for $\chi^{\mu\nu}$ includes many terms,

$$\chi^{\mu\nu} = a\, \partial_\gamma h^{\mu\nu} \partial_\beta h^{\gamma\beta} + b\, \partial_\gamma h^{\mu\beta} \partial_\beta h^{\nu\gamma} + \dots$$

Suppose we have found a $\chi^{\mu\nu}$ that satisfies $\partial_\mu (T^{\mu\nu} + \chi^{\mu\nu}) = 0$ to $\mathcal{O}(h \cdot T)$.

But now the linearized eq. of motion for $h_{\mu\nu}$ cannot be exactly satisfied, because $\partial_\mu T^{\mu\nu} = 0$ is a consequence of that equation.

So we ask for a new, nonlinear equation for $h_{\mu\nu}$ with $T^{\mu\nu}$ as its source.

The form of $\chi^{\mu\nu}$ is not completely determined by knowledge of $\partial_\mu \chi^{\mu\nu}$, so we can impose a further constraint that a local invariance like $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu + \mathcal{O}(\varepsilon h)$ leave the equation for $h_{\mu\nu}$ invariant.

Alternatively, we can insist that the equation for $h_{\mu\nu}$ be deduced by an action principle, which constrains the form of the equation sufficiently to determine $\chi^{\mu\nu}$ and the equation for $h_{\mu\nu}$ to $\mathcal{O}(h^2)$.

But now the new $T^{\mu\nu} + \chi^{\mu\nu}$ is only conserved to $\mathcal{O}(h \cdot T)$, so we wash, rinse, and repeat, extending the story to higher order in h.

This procedure works, but is unwieldy.

At this point, we had better enter Einstein's world, a world in which spacetime is the main character.

(Cf. S. Deser, "Self Interaction and Gauge Invariance," GR and Gravitation, 1, 9-18.)