Zee 3.6,
Feynman ch. 3,4

Dynamics of Gravity

The effects of gravitation on freely falling particles are encoded in the metric tensor $g_{\mu\nu}$. We have seen that in the Newtonian limit, $g_{\mu\nu}$ is related to the gravitational potential ϕ via $g_{00} = -(1+2\phi)$.

We now turn to the question of how to determine the dynamics of gravity, i.e. how to determine $g_{\mu\nu}(\vec{x}, t)$ given properties of gravitational sources.

Scalar Gravity?

One possibility is that even away from the Newtonian limit, gravity is entirely encoded in a potential $\phi(\vec{x}, t)$, for example with $g_{\mu\nu} = \eta_{\mu\nu}(1+2\phi)$.

Attempt 1: If ϕ satisfies the Newtonian relation $\nabla^2 \phi = 4\pi G \rho$, then changes in ρ affect ϕ instantaneously at a distance, in seeming conflict with special relativity.

This problem is easy to fix. Suppose that instead, the scalar potential (i.e. scalar field) $\phi(\vec{x}, t)$ satisfies

$$\boxed{-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = 4\pi G \rho}.$$

When $\rho = 0$ this is the wave equation for ϕ, which is consistent with special relativity ($\partial_\mu \partial^\mu \phi = 0$).

However, 2) $\phi(\vec{x},t)$ is a scalar field, while the mass density $\rho(\vec{x},t)$ transforms nontrivially under Lorentz transformations.

The density $\rho(\vec{x},t)$ is a component of a tensor in special relativity, the stress-energy tensor, which describes the density and flux of energy and momentum.

The stress-energy tensor in special relativity is the conserved current associated with spacetime translations via Noether's theorem. We denote the stress-energy tensor by $T_{\mu\nu}$. Conservation implies $\boxed{\partial_\mu T^{\mu\nu}=0,}$

i.e.
$$\frac{\partial}{\partial t}T^{0\nu}+\sum_{i=1}^{3}\frac{\partial}{\partial x^i}T^{i\nu}=0$$

As long as $T^{\mu\nu}$ falls off sufficiently quickly at ∞, the conservation law implies four time-independent quantities, one for each value of the index ν.

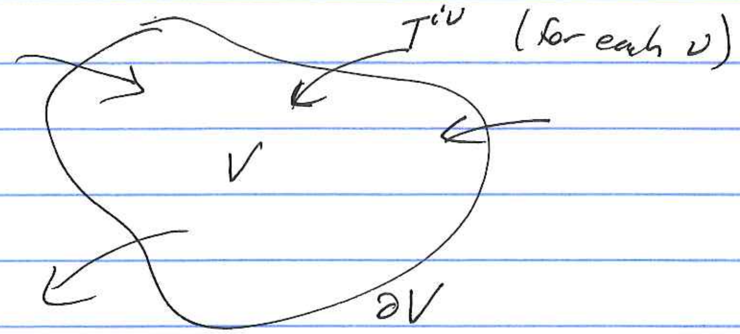
Consider
$$\int_V d^3x\left[\frac{\partial}{\partial t}T^{0\nu}+\sum_{i=1}^{3}\frac{\partial}{\partial x^i}T^{i\nu}\right]=0$$
, integrated over some region V

$$\frac{d}{dt}\int_V d^3x\, T^{0\nu}=-\int_V d^3x\sum_{i=1}^{3}\frac{\partial}{\partial x^i}T^{i\nu}$$

$$=-\int_{\partial V} d^2x\; n_i T^{i\nu}$$
, where ∂V is the boundary of V and n_i = unit vector normal to boundary of V.

(compare with $\int_V d^3x\, \nabla\cdot\vec{J}=\int_{\partial V} d^2x\, \hat{n}\cdot\vec{J}$ — Gauss' law)

The surface integral $\int_{\partial V} d^2x\, n_i T^{i\nu}$ represents the flux of $T^{i\nu}$ through the surface ∂V.



Taking the region V to fill all space, as long as $T^{i\nu}$ falls off quickly enough at ∂V,

$$\frac{d}{dt}\left(\int d^3x\, T^{0\nu}\right) = 0.$$

Defining $P^\nu \equiv \int d^3x\, T^{0\nu}$,

P^0 is the conserved quantity associated with time-translation invariance, i.e. the energy E. Hence, T^{00} is the energy density, $\boxed{T^{00} = \rho}$.

P^i, $i=1,2,3$ are the conserved quantities associated with spatial translation invariance in each of the three orthogonal directions, i.e. the spatial momentum $\vec{p}$. Hence, T^{0i} is the momentum density.

The components T^{ij} give the flux of the i^{th} component of momentum across a surface $x^j = \text{const}$. For a fluid, T^{12}, T^{13}, T^{23} are the components of the shear stress, and T^{11}, T^{22}, T^{33} the pressure.

In order to salvage the scalar theory of gravity, we could guess that the correct equation for $\phi(\vec{x},t)$ is

$$\boxed{\partial_\mu \partial^\mu \phi = -T^\mu{}_\mu \cdot 4\pi G}$$

-ve sign because $T^0{}_0 = -T^{00} = -\rho$.

This has the features that both sides of the equation are Lorentz scalars if ϕ is a scalar field, and as long as $|T^{00}| \gg |T^{ii}|$ for all $i = 1,2,3$, as is typically the case in the Newtonian limit, then the above equation for ϕ has the correct Newtonian limit.

The problem with this theory of gravity is 3) it doesn't agree with observation. This theory predicts incorrect bending of light by the sun, precession of the perihelion of Mercury, ...

At this stage we give up on our attempt to formulate a scalar theory of gravity.

Vector Gravity?

How about a vector field describing gravity, like in E&M?

Problem: 1) Like charges repel if the interaction is mediated by a vector field. In gravity it seems that things attract one another.

So, we give up on a vector theory of gravity.

Tensor Gravity

The next-simplest possibility is that gravity is mediated by a two-index tensor field $h_{\mu\nu}(\vec{x},t)$, say with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We can anticipate that the source for $h_{\mu\nu}$, i.e. the term replacing $\rho(\vec{x},t)$ on the right-hand side of the equation for $h_{\mu\nu}$, is the stress-energy tensor $T^{\mu\nu}$.

The symmetric and antisymmetric parts of $h_{\mu\nu}$ (as a matrix) are preserved by Lorentz transformations.

For example, if $h_{\mu\nu} = h_{\nu\mu}$, then

$$h'_{\alpha\beta} = (\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}\, h_{\mu\nu} = (\Lambda^{-1})^{\nu}{}_{\beta}(\Lambda^{-1})^{\mu}{}_{\alpha}\, h_{\nu\mu} = h'_{\beta\alpha}$$

Similarly, if $h_{\mu\nu} = -h_{\nu\mu}$, then

$$h'_{\alpha\beta} = (\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}\, h_{\mu\nu} = (\Lambda^{-1})^{\nu}{}_{\beta}(\Lambda^{-1})^{\mu}{}_{\alpha}\,(-h_{\nu\mu}) = -h'_{\beta\alpha}$$

(Belinfante-Rosenfeld stress-energy)

The stress-energy tensor can be chosen to be symmetric, $T_{\mu\nu} = T_{\nu\mu}$. Hence, we will guess that we only need the symmetric part of $h_{\mu\nu}$, and we assume that $h_{\mu\nu} = h_{\nu\mu}$. This is consistent with the interpretation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

To determine a reasonable equation for the dynamics of $h_{\mu\nu}$, we are guided by a few principles:

1) Lorentz covariance
2) Symmetry, e.g. $T_{\mu\nu} = T_{\nu\mu}$
3) Conservation $\partial_\mu T^{\mu\nu} = 0$
4) Simplicity: a) linear in $h_{\mu\nu}$ and its derivatives; b) at most 2 derivatives
5) Agrees with experiment and observation.

If the right-hand side of the equation is proportional to $T_{\mu\nu}$, the left-hand side should be composed of terms each of which is a symmetric rank-2 tensor.

Terms with the minimal number of derivatives are:

No derivatives: $h_{\mu\nu}$, $\eta_{\mu\nu} h_\alpha{}^\alpha \equiv \eta_{\mu\nu} h$

One derivative: none.

Two derivatives: $\partial_\alpha \partial^\alpha h_{\mu\nu}$

$\partial_\mu \partial_\nu h$

$\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}$

$\eta_{\mu\nu} \partial_\alpha \partial^\alpha h$

$\eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta}$

Suppose the equation for $h_{\mu\nu}$ has the form

$$\boxed{\begin{aligned} a\,\partial_\alpha \partial^\alpha h_{\mu\nu} + b\,\partial_\mu \partial_\nu h + c\left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}\right) \\ + d\,\eta_{\mu\nu} \partial_\alpha \partial^\alpha h + e\,\eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} = -\lambda T_{\mu\nu} \end{aligned}}$$

for some constants a, b, c, d, e, λ.

We don't add terms without derivatives because we have seen in our discussion of Newtonian gravity that if we modify the equation $\nabla^2 \phi = 4\pi G\rho$ to $\nabla^2\phi - m^2\phi = 4\pi G\rho$, then solutions fall exponentially away from a localized source ρ. Gravity is a long-ranged force, so we assume that "mass terms" linear in $h_{\mu\nu}$ vanish.

For particular relations between a, b, c, d, e, the conservation law $\partial^\mu T_{\mu\nu} = 0$ will follow as a consequence of the equation for $h_{\mu\nu}$.
Acting on the equation with ∂^μ, we get

$$a\,\partial_\alpha\partial^\alpha\partial^\mu h_{\mu\nu} + b\,\partial^\mu\partial_\mu\partial_\nu h + c\left(\partial^\mu\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial^\mu\partial_\nu\partial^\alpha h_{\alpha\mu}\right)$$
$$+\, d\,\partial_\nu\partial_\alpha\partial^\alpha h + e\,\partial_\nu\partial_\alpha\partial_\beta h^{\alpha\beta} = -\lambda\,\partial^\mu T_{\mu\nu} = 0.$$

$$\partial_\alpha\partial^\alpha\partial^\mu h_{\mu\nu}\,(a+c) + \partial^\mu\partial_\mu\partial_\nu h\,(b+d) + \partial^\mu\partial_\nu\partial^\alpha h_{\alpha\mu}\,(c+e) = 0.$$

This is automatically satisfied if $a = e = -c$ and $b = -d$.
Choosing $a = 1$, the equation for $h_{\mu\nu}$ becomes

$$\boxed{\begin{aligned}&\partial_\alpha\partial^\alpha h_{\mu\nu} - \left(\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu}\right) + \eta_{\mu\nu}\,\partial_\alpha\partial_\beta h^{\alpha\beta}\\ &\quad + b\left(\partial_\mu\partial_\nu h - \eta_{\mu\nu}\,\partial_\alpha\partial^\alpha h\right) = -\lambda\,T_{\mu\nu}\end{aligned}}$$

Gauge Invariance

Suppose we replace $h_{\mu\nu}$ with $h_{\mu\nu} + \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu$ for some vector field $\varepsilon_\mu(x)$.
The left-hand side of the equation for $h_{\mu\nu}$ changes by:

$$\partial_\alpha\partial^\alpha(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) - \partial_\mu\partial^\alpha(\partial_\alpha\varepsilon_\nu + \partial_\nu\varepsilon_\alpha) - \partial_\nu\partial^\alpha(\partial_\alpha\varepsilon_\mu + \partial_\mu\varepsilon_\alpha)$$
$$+\,\eta_{\mu\nu}\,\partial_\alpha\partial_\beta(\partial^\alpha\varepsilon^\beta + \partial^\beta\varepsilon^\alpha) + b\,\partial_\mu\partial_\nu(\partial_\alpha\varepsilon^\alpha + \partial_\alpha\varepsilon^\alpha)$$
$$-\,b\,\eta_{\mu\nu}\,\partial_\alpha\partial^\alpha(\partial_\beta\varepsilon^\beta + \partial_\beta\varepsilon^\beta)$$
$$= \partial_\alpha\partial^\alpha\partial_\mu\cancel{\varepsilon_\nu(1-1)} + \partial_\alpha\partial^\alpha\partial_\nu\cancel{\varepsilon_\mu(1-1)} + \partial_\mu\partial_\nu\partial_\alpha\varepsilon^\alpha(-1-1+2b)$$
$$+\,\eta_{\mu\nu}\,\partial_\alpha\partial^\alpha\partial_\beta\varepsilon^\beta(1+1-2b)$$
$$= 0 \quad \text{if } b = 1.$$

With the choice $b=1$, the equation for $h_{\mu\nu}$ is invariant under $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$.

For any solution to the equation for $h_{\mu\nu}$, there are then a continuum of other solutions, one for every $\varepsilon_\mu(x)$.

This is a redundancy in the description of the field $h_{\mu\nu}$, a gauge invariance much like in the relativistic description of electromagnetism.

In E&M, the equation for the vector field A_μ is

$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 4\pi J_\nu$, where J_ν is the 4-vector current.

The term in parentheses is the field strength tensor

$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. It is antisymmetric, and its nonvanishing components are the components of $\vec{E}$ and $\vec{B}$. (Note that $\partial^\nu J_\nu = 0$ follows from the eqn. for A_μ.)

The equation of motion for A_μ is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta(x)$ for any function $\theta(x)$.

This is a redundancy in the description of the electromagnetic field in terms of A_μ, and a gauge condition is required in order to uniquely specify a solution for A_μ.

The redundancy reduces the number of propagating degrees of freedom in A_μ. There are 4 components in A_μ, but only 2 propagating degrees of freedom, the two helicities of the photon.

If we can describe gravitation with fewer degrees of freedom, then we are compelled to at least try.

We are thereby led to the <u>linearized Einstein equations</u>:

$$\boxed{\partial_\alpha \partial^\alpha h_{\mu\nu} - (\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}) + \eta_{\mu\nu}\partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h - \eta_{\mu\nu}\partial_\alpha \partial^\alpha h = -\lambda T_{\mu\nu}}$$

Just as in electrodynamics, we can choose to fix a gauge condition on $h_{\mu\nu}$ in order to make equations look simpler.

For example, we may insist that $h_{\mu\nu}$ satisfy,

$$\boxed{\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h} \quad \text{Harmonic gauge.}$$

To see that we can make this choice, assume

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h = f^\nu(x) \neq 0.$$

Let $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ (gauge transformation)

Then $\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h \to \underbrace{\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h}_{f_\nu(x)} + \partial_\mu\partial^\mu \xi_\nu + \cancel{\partial_\mu\partial_\nu \xi^\mu} - \cancel{\frac{1}{2}\cdot 2\partial_\nu\partial_\mu \xi^\mu}$

$$= 0 \quad \text{if} \quad \partial_\mu\partial^\mu \xi_\nu = -f_\nu(x),$$

which can be solved for $\xi_\nu(x)$.

Note that the harmonic gauge condition does not completely fix the gauge, because we can make a further gauge transformation with $\partial_\mu\partial^\mu \xi^\nu = 0$ (the wave equation) while maintaining the harmonic gauge condition.

In the harmonic gauge, the linearized Einstein equations become:

$$\partial_\alpha\partial^\alpha h_{\mu\nu} - \cancel{\frac{1}{2}\partial_\mu\partial_\nu h} - \cancel{\frac{1}{2}\partial_\nu\partial_\mu h} + \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial^\alpha h + \cancel{\partial_\mu\partial_\nu h} - \eta_{\mu\nu}\partial_\alpha\partial^\alpha h = -\lambda T_{\mu\nu}$$

or,

$$\boxed{\partial_\alpha\partial^\alpha h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial^\alpha h = -\lambda T_{\mu\nu}}$$

linearized Einstein equations in harmonic gauge.

We can simplify the appearance of the linearized Einstein equations still further.

Given a 2-index tensor $A_{\mu\nu}$, define

$$\boxed{\bar{A}_{\mu\nu} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{2}\eta_{\mu\nu} A^{\sigma}{}_{\sigma}}$$

For a symmetric tensor like $h_{\mu\nu}$,

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h}$$

Also note, $\bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$. (Exercise)

The linearized Einstein eqs. in harmonic gauge may be written

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\lambda T_{\mu\nu}}$$

Taking the trace: $\partial_\alpha \partial^\alpha \left(h^{\mu}{}_{\mu} - \frac{1}{2}\delta^{\mu}{}_{\mu} h\right) = -\lambda T^{\mu}{}_{\mu}$

or, $-\partial_\alpha \partial^\alpha h = -\lambda T$

Adding $\frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial^\alpha h$ to the linearized Einstein eqs gives

$$\partial_\alpha \partial^\alpha h_{\mu\nu} = -\lambda T_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial^\alpha h = -\lambda T_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\lambda T$$

i.e.,

$$\boxed{\partial_\alpha \partial^\alpha h_{\mu\nu} = -\lambda \bar{T}_{\mu\nu}}$$

The Gravitational Coupling λ

In the harmonic gauge, the linearized Einstein equations took the form,

$$\boxed{\partial_\alpha \partial^\alpha h_{\mu\nu} = -\lambda\left(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} T\right)},$$

but we still need to identify the constant λ, which determines the strength of the gravitational coupling to $T_{\mu\nu}$.

In the Newtonian limit, $T^{00} = T_{00} = \rho$, $T^\mu{}_\mu \approx T^0{}_0 = -\rho$,

$$\boxed{\partial_\alpha \partial^\alpha h_{00} \approx -\lambda\left(\rho - \tfrac{1}{2}(-1)(-\rho)\right) = -\frac{\lambda}{2}\rho}$$

Earlier, by comparing the geodesic equation for a freely falling particle in the Newtonian limit to $\vec{a} = -\nabla\phi$, we deduced $h_{00} = -2\phi$, and $\nabla^2\phi = 4\pi G\rho$.
For static h_{00}, we have

$$\left.\begin{aligned} \nabla^2 h_{00} &= -2\nabla^2\phi = -8\pi G\rho \\ &= -\frac{\lambda}{2}\rho \end{aligned}\right\} \boxed{\lambda = 16\pi G}$$

With this identification of the coupling λ, we now have completely determined the linearized Einstein equations, which in arbitrary gauge are,

$$\boxed{\begin{aligned} \partial_\alpha \partial^\alpha h_{\mu\nu} - \left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}\right) + \eta_{\mu\nu}\partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h - \eta_{\mu\nu}\partial_\alpha \partial^\alpha h \\ = -16\pi G\, T_{\mu\nu} \end{aligned}}$$