Weinberg 12.4

An Action Principle for Gravity

The gravitational action should be invariant under coordinate transformations and should be composed of terms with two derivatives of the metric and its inverse.

The invariant volume element is $\sqrt{|g|}\, d^4x$, and the only scalar that suits the bill is R. Hence, we will study the equations that follow by varying the action $S = S_M + S_G$, with S_M the matter action, and

$$\boxed{S_G = -\frac{1}{16\pi G}\int d^4x \sqrt{|g|}\, R}$$

Taking $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, using $R = g^{\mu\nu} R_{\mu\nu}$,

$$\delta\left(\sqrt{|g|}\, R\right) = \sqrt{|g|}\, R_{\mu\nu}\, \delta g^{\mu\nu} + R\, \delta\sqrt{|g|} + \sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu}.$$

From the definition of $R_{\mu\nu}$ in terms of $\Gamma^\lambda_{\mu\nu}$, and $\Gamma^\lambda_{\mu\nu}$ in terms of $g_{\mu\nu}$, it is straightforward to show the <u>Palatini identity</u>:

$$\boxed{\delta R_{\mu\nu} = \left(\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}}$$

Hence $\sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu} = \sqrt{|g|}\left[\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}\right]$

Using $V^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\, \partial_\mu\left(\sqrt{|g|}\, V^\mu\right)$,

$$\sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu} = \partial_\nu\left(\sqrt{|g|}\, g^{\mu\nu}\, \delta\Gamma^\lambda_{\mu\lambda}\right) - \partial_\lambda\left(\sqrt{|g|}\, g^{\mu\nu}\, \delta\Gamma^\lambda_{\mu\nu}\right)$$

Hence, $\int d^4x \sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu} = 0.$

Using $\delta \ln \det M = \operatorname{Tr} M^{-1} \delta M$ with $M = g_{\mu\nu}$,

$$\boxed{\delta\sqrt{|g|} = \tfrac{1}{2}\sqrt{|g|}\, g^{\mu\nu}\, \delta g_{\mu\nu}}$$

Using $\delta(g^{\mu\nu} g_{\nu\lambda}) = 0$, $\boxed{\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma}\, \delta g_{\rho\sigma}}$

Hence, $\boxed{\delta S_G = \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\left[R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R\right]\delta g_{\mu\nu}}$

The energy-momentum tensor for matter may be defined in terms of the variation of the matter action with respect to the metric:

$$\boxed{\delta S_M = \tfrac{1}{2}\int d^4x \sqrt{|g|}\, T^{\mu\nu}\, \delta g_{\mu\nu}}$$

Hence, $\delta S = \delta S_G + \delta S_M$

$$= \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\, \delta g_{\mu\nu}\left[R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + 8\pi G T^{\mu\nu}\right]$$

$\delta S = 0$ gives the equation of motion, which we recognize as the Einstein equation:

$$\boxed{R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu}}$$

The Matter Action

Example: Scalar field

Suppose a scalar field $\phi(x)$ is described by the action

$$S_m = \int d^4x \sqrt{|g|} \underbrace{\left\{ -\frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, g^{\mu\nu} - V(\phi) \right\}}_{\mathcal{L}}$$

where the potential $V(\phi)$ depends on ϕ but not its derivatives.

In classical mechanics, stationarizing the action under variation of trajectories $q(t)$ leads to the Euler-Lagrange equation,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

Similarly, stationarizing S_m with respect to variations of $\phi(x)$ leads to the equation of motion for $\phi(x)$,

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial}{\partial (\partial_\mu \phi)} \left(\sqrt{|g|} \mathcal{L} \right) \right) = \frac{\partial}{\partial \phi} \left(\sqrt{|g|} \mathcal{L} \right), \text{ or}$$

$$-\partial_\mu \left(\sqrt{|g|} (\partial_\nu \phi) g^{\mu\nu} \right) = -\sqrt{|g|} \frac{\partial V}{\partial \phi}, \text{ or}$$

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} (\partial_\nu \phi) g^{\mu\nu} \right) = \frac{\partial V}{\partial \phi}, \text{ or}$$

$$\boxed{D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi}}.$$

If $V = 0$ this is the wave equation for ϕ in curved spacetime.

The energy-momentum tensor for the scalar field is

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g_{\mu\nu}} \quad , \text{ or}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}} .$$

Using $\delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|}\, g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2} \sqrt{|g|}\, g_{\mu\nu} \delta g^{\mu\nu}$, we have

$$\boxed{T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - g_{\mu\nu} \mathcal{L}}$$

The Vielbein Formalism

In order to turn a special-relativistic equation into a general-relativistic equation including gravity, we need only replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and derivatives ∂_μ to covariant derivatives D_μ.
This works for equations involving tensors, but we have not constructed a covariant derivative $D_\mu \Psi$ for spinor fields Ψ.

As a hint of the difficulty with spinors in general relativity, it is helpful to know some things about spinors in special relativity. All we really need to know is that a crucial part of the relativistic theory of spinors is the Dirac γ-matrices, 4x4 matrices γ^a, $a \in \{0,1,2,3\}$, which satisfy the relation

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$$

We might think that in general relativity spinors would then depend on matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} \stackrel{?}{=} 2g^{\mu\nu}$$

But $g^{\mu\nu}$ is a function of x^α, so γ^μ would need to be a matrix that depends on x^α. How would we solve a different matrix equation at each point in spacetime?

This puzzle motivates a different approach to determine the effects of gravitation, an approach still grounded in the Principle of Equivalence.

At each point in spacetime there are locally inertial coordinates ξ^a. In the neighborhood of that point the spacetime metric takes the special relativistic form,

$$ds^2 = \eta_{ab}\, d\xi^a\, d\xi^b.$$

In a general coordinate system, the metric becomes

$$ds^2 = \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} dx^\mu dx^\nu$$

$$\equiv g_{\mu\nu}\, dx^\mu dx^\nu.$$

Hence, $$g_{\mu\nu} = \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu}$$

$$\boxed{g_{\mu\nu} \equiv \eta_{ab}\, e_\mu{}^a\, e_\nu{}^b}$$

where $e_\mu{}^a \equiv \dfrac{\partial \xi^a}{\partial x^\mu}$ is called a tetrad or vierbein in four spacetime dimensions, or a vielbein more generally.

We think of $e_\mu{}^a$ as a set of 4 covariant vector fields $e_\mu{}^0$, $e_\mu{}^1$, $e_\mu{}^2$, and $e_\mu{}^3$.

Given a contravariant 4-vector V^μ, the vierbein transforms the vector to locally inertial coordinates:

$$V^a \equiv e_\mu{}^a V^\mu ,$$

which we think of as a set of four scalars V^0, V^1, V^2, V^3.

We can construct an inverse vielbein which transforms covariant vectors to locally inertial coordinates at a point:

$$g_{\mu\sigma} = \eta_{ab}\, e_\mu{}^a e_\sigma{}^b$$

$$\delta_\mu^\nu = g^{\nu\sigma} g_{\mu\sigma} = e_\mu{}^a \underbrace{e_\sigma{}^b \eta_{ab}\, g^{\nu\sigma}}_{(e^{-1})_a{}^\nu \equiv e_a{}^\nu}$$

(note the placement of Lorentz index a and general coordinate index ν to distinguish $e_\nu{}^a$ and $e_a{}^\nu$.)

Because $e_a{}^\nu$ is the inverse of $e_\mu{}^a$, we also have

$$e_a{}^\mu e_\mu{}^b = \delta_a{}^b$$

$$\left(e_\sigma{}^c \eta_{ac}\, g^{\mu\sigma}\right) e_\mu{}^b = \delta_a{}^b$$

$$\times\, \eta^{ad}: \quad \boxed{e_\sigma{}^d e_\mu{}^b g^{\mu\sigma} = \eta^{db}}$$

Similarly, $\boxed{e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}}$ (Exercise)

By always referring to locally inertial coordinates, tensors are reduced to sets of scalars under coordinate transformations. We can also refer spinors to locally inertial coordinates, and γ-matrices satisfying

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}$$ are again relevant.

In this formalism, equations describing interactions with gravitation need to satisfy two conditions:

1) The equation is generally covariant, with all fields treated as scalars <u>except</u> the vierbein $e_\mu{}^a$.

2) The equation is covariant under independent Lorentz transformations at each point.
   For example $V^a(x) \to \Lambda^a{}_b(x) V^b(x)$, etc.

The second requirement demands a new type of covariant derivative $D_a = e_a{}^\mu \underbrace{\left(\frac{\partial}{\partial x^\mu} + \Gamma_\mu(x)\right)}_{D_\mu}$,

where $\Gamma_\mu(x)$ is a matrix depending on what type of tensor or spinor field it acts on, chosen so that D_μ acting on an object transforms under local Lorentz transformations the same way as the object, cancelling nonhomogeneous terms like $\frac{\partial}{\partial x^\mu} \Lambda^a{}_b(x)$.

$\Gamma_\mu(x)$ is called the <u>spin connection</u>.

Example: Covariant derivative of covariant vector:

$$(D_\mu V)_c = \partial_\mu V_c + (\Gamma_\mu)^d{}_c V_d \text{ , where}$$

Γ_μ acting on a covariant vector has the form

$$(\Gamma_\mu)^d{}_c = e_a^\lambda \left(\partial_\mu e_{\lambda b} - \Gamma^\sigma_{\mu\lambda} e_{\sigma b}\right) (\Sigma^{ab})_c{}^d$$

Christoffel symbols

where $e_{\lambda b} \equiv \eta_{bc} e_\lambda{}^c$ and

$$(\Sigma^{ab})_c{}^d = \tfrac{1}{2}\left(\delta^a{}_c \eta^{bd} - \delta_c^b \eta^{ad}\right)$$

(= Lorentz generator in vector representation)

After some tedious algebra, one can check that
1) $(D_\mu V)_c$ transforms as a vector under local Lorentz transformations, and
2) $(D_\mu V)_c$ agrees with the usual covariant derivative:

$$(D_\mu V)_\nu \equiv e_\nu{}^c (D_\mu V)_c = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu} V_\sigma$$

In d spacetime dimensions, it would seem that $e_\mu{}^a$ has $d \times d = d^2$ independent components. However, $\frac{d(d-1)}{2}$ local Lorentz transformations can restrict the same # of components.
Hence, there are $d^2 - \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$ independent components of the vielbein up to local Lorentz transfs, the same # of components as in $g_{\mu\nu}$.

Appendix:

Spin Connection vs Christoffel Symbols

$$(D_\mu V)_C = \partial_\mu V_C + (\Gamma_\mu)^D{}_C V_D \quad , \quad \begin{array}{l} V = \text{Lorentz vector (covariant)} \\ \Gamma_\mu = \text{Spin Connection} \end{array}$$

$$(D_\mu V)_C = \partial_\mu V_C + \underbrace{e^\lambda{}_A(\partial_\mu e_{\lambda B} - \Gamma^\sigma_{\mu\lambda} e_{\sigma B})}_{\omega_{\mu AB}} (\Sigma^{AB})_C{}^D V_D$$

Christoffel symbols

$$(\Sigma^{AB})_C{}^D = \tfrac{1}{2}(\delta^A_C \eta^{BD} - \delta^B_C \eta^{AD})$$

= Lorentz generator in vector representation

$$(D_\mu V)_C = \partial_\mu V_C + \tfrac{1}{2}\Big(e^\lambda{}_C \partial_\mu e_\lambda{}^D - e^{\lambda D}\partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\lambda} e^\lambda{}_C e_\sigma{}^D + \Gamma^\sigma_{\mu\lambda} e^{\lambda D} e_{\sigma C}\Big) V_D$$

$$= \partial_\mu V_C + \tfrac{1}{2}\Big(e^\lambda{}_C \partial_\mu e_\lambda{}^D - e^{\lambda D}\partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\lambda} e^\lambda{}_C e_\sigma{}^D + \tfrac{1}{2} e^{\lambda D} e_{\sigma C} g^{\sigma\gamma}(\partial_\mu g_{\lambda\gamma} + \partial_\lambda g_{\mu\gamma} - \partial_\gamma g_{\mu\lambda})\Big) V_D$$

$$= \partial_\mu V_C + \tfrac{1}{2}\Big(e^\lambda{}_C \partial_\mu e_\lambda{}^D - e^{\lambda D}\partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\lambda} e^\lambda{}_C e_\sigma{}^D + \tfrac{1}{2} e^{\lambda D} e^\gamma{}_C (\partial_\mu g_{\lambda\gamma} + \partial_\lambda g_{\mu\gamma} - \partial_\gamma g_{\mu\lambda})\Big) V_D$$

$$e^C_\nu (D_\mu V)_C = e_\nu{}^C \partial_\mu V_C + \tfrac{1}{2}\Big(\partial_\mu e_\nu{}^D - e^{\lambda D} e_\nu{}^C \partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\nu} e_\sigma{}^D + \tfrac{1}{2} e^{\lambda D}(\partial_\mu g_{\lambda\nu} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda})\Big) V_D$$

$$e_\nu{}^C (D_\mu V)_C = e_\nu{}^C \partial_\mu V_C + \frac{1}{2}\Big(\partial_\mu e_\nu{}^D - e^{\lambda D}(\partial_\mu g_{\nu\lambda} - e_{\lambda C}\partial_\mu e_\nu{}^C)$$

$$- \Gamma^\sigma_{\mu\nu} e_\sigma{}^D + \frac{1}{2} e^{\lambda D}(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda})\Big) V_D$$

$$= e_\nu{}^C \partial_\mu V_C + \frac{1}{2}\Big(\partial_\mu e_\nu{}^D + \partial_\mu e_\nu{}^D - \Gamma^\sigma_{\mu\nu} e_\sigma{}^D$$

$$+ \frac{1}{2} e^{\lambda D}(-\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda})\Big) V_D$$

$$= e_\nu{}^C \partial_\mu V_C + (\partial_\mu e_\nu{}^D) V_D - \Gamma^\sigma_{\mu\nu} e_\sigma{}^D V_D$$

$$= \partial_\mu V_\nu - \cancel{V_C \partial_\mu e_\nu{}^C} + \cancel{(\partial_\mu e_\nu{}^D) V_D} - \Gamma^\sigma_{\mu\nu} V_\sigma$$

$$\boxed{(D_\mu V)_\nu \equiv e_\nu{}^C (D_\mu V)_C = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu} V_\sigma}$$