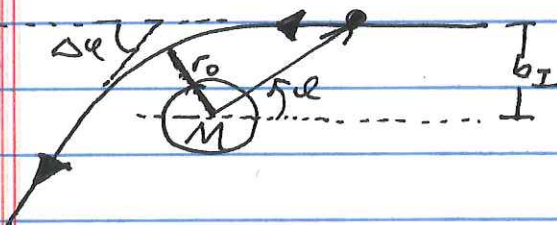


Bending of Light by a Massive Object



r_0 = value of r at closest approach ($t=0$)

b_I = impact parameter

$$\left(\frac{dr}{dt}\right)^2 + \left(1 - \frac{2GM}{r}\right) \frac{\tilde{L}^2}{r^2} = \tilde{E}^2$$

$$r^2 \frac{d\varphi}{dt} = \tilde{L}$$

$$\Rightarrow \frac{d\varphi}{dr} = \pm \frac{\tilde{L}}{r^2} \left[\tilde{E}^2 - \left(1 - \frac{2GM}{r}\right) \frac{\tilde{L}^2}{r^2} \right]^{-1/2}$$

Define $b \equiv \frac{\tilde{L}}{\tilde{E}} = \left(r^2 \frac{d\varphi}{dt} \right) / \left(\frac{dr}{dt} \right)$ as $r \rightarrow \infty$

Early times:



$$y = b_I$$

$$x = -t = r \cos \varphi$$

$$r \approx |t|$$

$$\tan \varphi = \frac{b_I}{x} \approx -\frac{b_I}{t}$$

$$\frac{d}{dt} : \sec^2 \varphi \frac{d\varphi}{dt} = \frac{b_I}{t^2}, \quad \sec^2 \varphi t^2 = r^2$$

$$\Rightarrow r^2 \frac{d\varphi}{dt} = b_I$$

$$b = \frac{\tilde{L}}{\tilde{E}} = r^2 \frac{d\varphi}{dt} = b_I$$

$$\Rightarrow \frac{d\varphi}{dr} = \pm \left[\frac{r^4}{b^2} - (r - 2GM)r \right]^{-1/2}$$

Change in \mathcal{U} along trajectory:

$$\mathcal{U}_f - \mathcal{U}_i = \int d\mathcal{U} = 2 \int_{r_0}^{\infty} dr \left[\frac{r^4}{b^2} - r(r - 2GM) \right]^{-1/2}$$

At closest approach $r=r_0$, $\frac{dr}{d\tau} = 0$

$$\left(\frac{dr}{d\tau} \right)^2 + \left(1 - \frac{2GM}{r} \right) \frac{\tilde{L}^2}{r^2} = E^2$$

$$\left(1 - \frac{2GM}{r_0} \right) \frac{\tilde{L}^2}{r_0^2} = E^2$$

$$\frac{1}{r_0^2} \left(1 - \frac{2GM}{r_0} \right) = \frac{E^2}{\tilde{L}^2} = \frac{1}{b^2}$$

$$\mathcal{U}_f - \mathcal{U}_i = 2 \int_{r_0}^{\infty} dr \left[\frac{r^4}{r_0^2} \left(1 - \frac{2GM}{r_0} \right) - r^2 + 2GM/r \right]^{-1/2}$$

Let $u = 1/r$:

$$\mathcal{U}_f - \mathcal{U}_i = 2 \int_0^{1/r_0} du \left[\left(1 - \frac{2GM}{r_0} \right) \frac{1}{r_0^2} - u^2 + 2GM u^3 \right]^{-1/2}$$

$$\text{If } M=0, \quad \mathcal{U}_f - \mathcal{U}_i = 2 \int_0^{1/r_0} du \left[\frac{1}{r_0^2} - u^2 \right]^{-1/2} = \pi \quad (\text{Exercise})$$

If $M \neq 0$, but $\frac{2GM}{r_0} \ll 1$:

$$\Delta \mathcal{U} = (\mathcal{U}_f - \mathcal{U}_i) - \pi$$

$$= 2 \int_0^{1/r_0} du \left[\left(\frac{1}{r_0^2} - u^2 \right) - 2GM \left(\frac{1}{r_0^3} - u^3 \right) \right]^{-1/2} - \pi$$

$$\approx 2 \int_0^{1/r_0} du \left[\frac{1}{r_0^2} - u^2 \right]^{-1/2} + 2GM \int_0^{1/r_0} du \frac{\left(\frac{1}{r_0^3} - u^3 \right)}{\left(\frac{1}{r_0^2} - u^2 \right)^{3/2}} - \pi$$

$$\Delta\varphi \approx 2GM \int_0^{1/r_0} \frac{du}{\left(\frac{1}{r_0^2} - u^2\right)^{3/2}} \left[\left(\frac{1}{r_0^2} - u^2\right) \left(\frac{1}{r_0} + u\right) + \frac{u^2}{r_0} - \frac{u}{r_0^2} \right]$$

$$= 2GM \int_0^{1/r_0} du \left\{ \frac{\left(\frac{1}{r_0} + u\right)}{\left(\frac{1}{r_0^2} - u^2\right)^{3/2}} + \frac{\left(\frac{u^2}{r_0} - \frac{u}{r_0^2}\right)}{\left(\frac{1}{r_0^2} - u^2\right)^{3/2}} \right\}$$

Let $u = \frac{1}{r_0} \sin\theta$, $du = \frac{1}{r_0} \cos\theta d\theta$

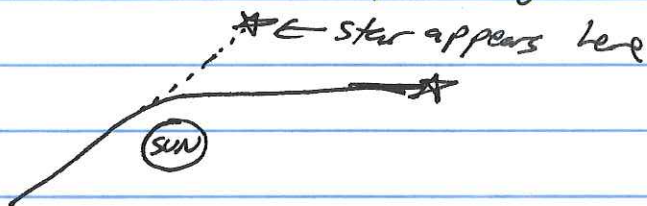
$$\frac{1}{r_0^2} - u^2 = \frac{1}{r_0^2} \cos^2\theta$$

$$\Delta\varphi \approx \frac{2GM}{r_0} \int_0^{\pi/2} d\theta \left\{ \frac{\cos\theta (1 + \sin\theta)}{\cos^3\theta} + \frac{\cos\theta (\sin^2\theta - \sin\theta)}{(\cos\theta)^3} \right\}$$

$$= \frac{2GM}{r_0} \int_0^{\pi/2} d\theta \left\{ \sin\theta + \frac{(1 - \sin\theta)}{\cos^2\theta} \right\}$$

$$= \frac{2GM}{r_0} (-\cos\theta) \Big|_0^{\pi/2} + \frac{2GM}{r_0} \frac{(\sin\theta - 1)}{\cos\theta} \Big|_0^{\pi/2}$$

$$\Delta\varphi \approx \frac{4GM}{r_0}$$



Deflection of starlight by sun:

$$r_0 = R_\odot = 6.96 \times 10^5 \text{ km}$$

$$GM_\odot = 1.48 \text{ km}$$

$$\Rightarrow \Delta\varphi = (8.5 \times 10^{-6} \text{ rad}) \times \frac{360^\circ}{2\pi} \cdot \frac{3600''}{1^\circ}$$

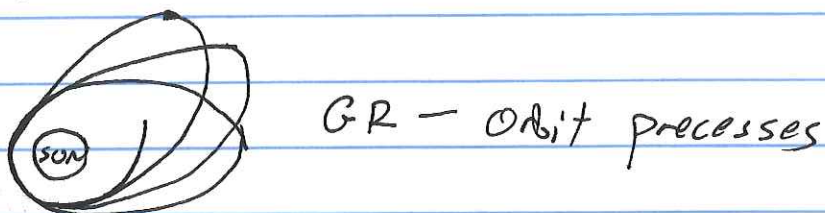
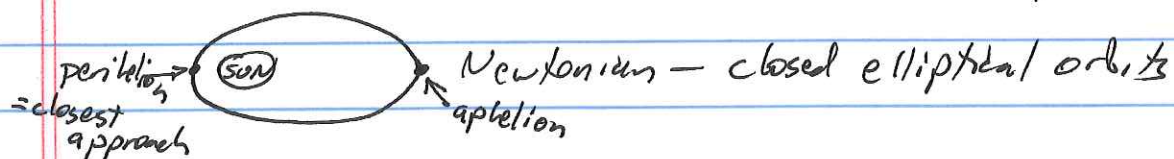
$$= 1.75'' \text{ arc deflection.}$$

In 1919, Arthur Stanley Eddington led an expedition to the island of Principe in the Gulf of Guinea to photograph stars in the vicinity of the sun during the May 29 solar eclipse. A second expedition, led by Andrew Crommelin went to Sobral, in Brazil. Their observations confirmed the magnitude of the bending of light predicted by Einstein, and propelled Einstein to super-fame.



Eddington → Einstein Telegram

Precession of the Perihelion of Planetary Orbits.



$$\left(\frac{dr}{dt}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2GM}{r}\right) \left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

$$r^2 \frac{d\varphi}{dt} = \tilde{L}$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left[\tilde{E}^2 - \left(1 - \frac{2GM}{r}\right) \left(1 + \frac{\tilde{L}^2}{r^2}\right) \right] \frac{r^4}{\tilde{L}^2}$$

$$u = \frac{1}{r}:$$

$$\boxed{\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{\tilde{L}^2} \left[\tilde{E}^2 - (1 - 2GMu) (1 + \tilde{L}^2 u^2) \right]}$$

Analogous Newtonian problem: ignore u^3 term.

$$\left(\frac{du}{d\varphi}\right)^2 = -q_0 + q_1 u - u^2, \text{ where } q_0, q_1 \text{ depend on } \tilde{E}, \tilde{L}, GM.$$

$$\text{Let } u = y + \frac{q_1}{2} \quad (\text{complete the square})$$

$$\left(\frac{dy}{d\varphi}\right)^2 = \left(-q_0 + \frac{q_1^2}{4} - \frac{q_1^2}{4}\right) - y^2 = \left(-q_0 + \frac{q_1^2}{4}\right) - y^2$$

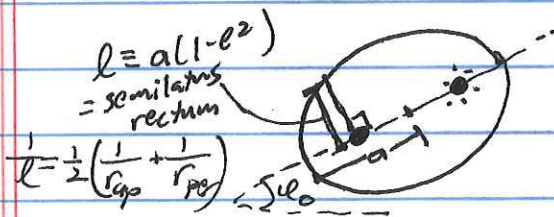
Solution: $y = \left(-q_0 + \frac{q_1^2}{4}\right)^{1/2} \cos(\psi - \psi_0)$

$$u = \frac{q_1}{2} + \left(-q_0 + \frac{q_1^2}{4}\right)^{1/2} \cos(\psi - \psi_0)$$

$$\frac{1}{r} = \frac{q_1}{2} + \left(-q_0 + \frac{q_1^2}{4}\right)^{1/2} \cos(\psi - \psi_0)$$

$$\frac{1}{r} = \frac{q_1}{2} \left[1 + \left(\frac{-4q_0}{q_1^2} + 1\right)^{1/2} \cos(\psi - \psi_0) \right]$$

This is an ellipse, $r = \frac{a(1-e^2)}{1+e \cos(\psi - \psi_0)}$



$$e = \text{eccentricity} = \frac{r_{ap} - r_{per}}{r_{ap} + r_{per}}$$

$$a = \text{semimajor axis} = \frac{r_{ap} + r_{per}}{2}$$

$$r_{ap} = (1+e)a, \quad r_{per} = (1-e)a$$

In our case, $e = \left(1 - \frac{4q_0}{q_1^2}\right)^{1/2}$

$$a = \frac{q_1}{2q_0}$$

$$l = \frac{q_1}{2q_0} \left(\frac{4q_0}{q_1^2}\right) = \frac{2}{q_1}$$

Correction to orbit due to GR due to u^3 term:

$$\begin{aligned} \left(\frac{du}{de}\right)^2 &= \frac{\tilde{E}^2 - 1}{\tilde{L}^2} + \frac{2GM}{\tilde{L}^2} u - u^2 + 2GM u^3 \\ &= -q_0 + q_1 u - u^2 + q_3 u^3 \\ &\quad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \frac{1-\tilde{E}^2}{\tilde{L}^2} & \frac{2GM}{\tilde{L}^2} & 2GM \end{array} \end{aligned}$$

Let $u = u_0 + y$, choose u_0 to eliminate linear term in y
 $\rightarrow q_1 - 2u_0 + 3q_3 u_0^2 = 0$

$$\left(\frac{dy}{de}\right)^2 = [-q_0 + q_1 u_0 - u_0^2 + q_3 u_0^3] + [-1 + 3q_3 u_0] y^2 + q_3 y^3$$

Circular orbit: $y = 0$; $r = \frac{1}{u_0} = \text{const.}$

- solution if $-q_0 + q_1 u_0 - u_0^2 + q_3 u_0^3 = 0$

Nearly circular orbit: y small \rightarrow neglect y^3 piece.

$$\left(\frac{dy}{de}\right)^2 \approx (\text{const}) - (1 - 3q_3 u_0) y^2$$

$$\begin{aligned} \text{solution: } y &= (\text{const})^{1/2} \cos\left(\sqrt{1 - 3q_3 u_0} (e - e_0)\right) \\ &= (\text{const})^{1/2} \cos\left(\sqrt{1 - \frac{6GM}{r_0}} (e - e_0)\right) \end{aligned}$$

$r_0 = \frac{1}{u_0} \approx \frac{2}{q_1}$ from Newtonian approximation.

$= \ell$ (semilatus rectum)

$\frac{1}{\ell} = \frac{1}{2} \left(\frac{1}{r_{\text{ap}}} + \frac{1}{r_{\text{per}}} \right) \approx \frac{1}{r_{\text{ap}}}$ for nearly circular orbit

Perihelion = closest approach

→ r minimum → y maximum

$$\cos\left(\sqrt{1 - \frac{6GM}{r_0}} (\varphi - \varphi_0)\right) \text{ maximum}$$

$$\rightarrow \varphi - \varphi_0 = 0, \frac{2\pi}{\sqrt{1 - \frac{6GM}{r_0}}} \approx 2\pi \left(1 + \frac{3GM}{r_0}\right), \text{ etc.}$$

r returns to r_0 if $\varphi \rightarrow \varphi + 2\pi + \frac{6\pi GM}{r_0}$

Precession of perihelion:

$$\Delta\varphi = \frac{6\pi GM}{r_0} \frac{\text{radians}}{\text{revolution}}$$

Mercury: $r_0 = 5.55 \times 10^7 \text{ km}$

$$GM_{\odot} = 1.47 \text{ km}$$

$$\rightarrow \Delta\varphi = 4.99 \times 10^{-7} \text{ rad/revolution}$$

$$1 \text{ revolution} = 0.24 \text{ years}$$

$$\rightarrow \Delta\varphi = 43.0'' \text{ arc/century} \quad (\text{Exercise})$$

Newtonian perturbations (due to Jupiter, etc.):

$$\Delta\varphi_N = 5557.62 \pm 0.20'' \text{ /century}$$

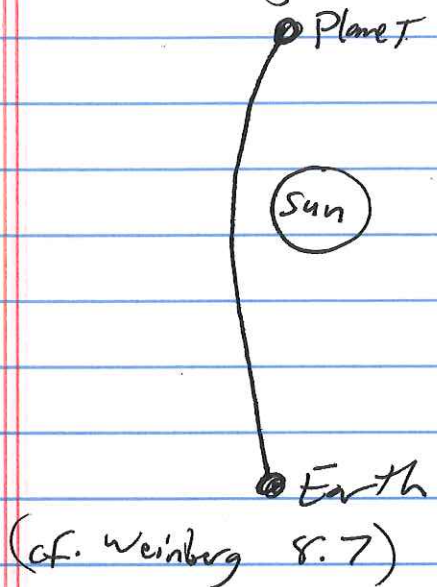
$$\text{Observed: } \Delta\varphi_{\text{obs}} = 5600.73 \pm 0.41'' \text{ /century}$$

$$\text{GR contribution: } \Delta\varphi_{\text{obs}} - \Delta\varphi_N = 43.1'' \text{ /century}$$

- agrees w/ prediction!

Radar Echo Delay

It is worth mentioning another test of GR based on gravitational time delay.



A light signal is reflected off of another planet, and the time of return is measured. GR predicts the time delay of the signal as a function of Sun's mass, Earth's radius, planet's radius, distance of closest approach, - Shapiro 1968, 1971

Targets other than planets may also be used, e.g. artificial satellites which reemit signals (Anderson 1975), anchored spacecraft in orbit around planet or landed on planet (Anderson, Shapiro, Rosenberg 1979)