Coordinate Singularities

Wald 6.4

In the Schwarzschild spacetime curvature invariants like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ behave regularly around $r=2GM$, but the metric depends on $\left(1-\frac{2GM}{r}\right)^{1/2}$ which behaves singularly.

This is an example of a coordinate singularity, which can be eliminated by a change of coordinates. A simple analogy is provided by the metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2, \quad \begin{array}{c} -\infty < x < \infty \\ 0 < t < \infty \end{array}$$

A change of coordinates $t \to t' = 1/t$ removes the singularity at $t=0$:

$$ds^2 = (t')^4\left(\frac{-dt'}{(t')^2}\right)^2 + dx^2$$

$$= -(dt')^2 + dx^2 \qquad \text{Minkowski space.}$$

The region covered by the original coordinates $0<t<\infty$ is the upper-half plane in Minkowski space, $0<t'<\infty$.

The spacetime described by the original metric is <u>geodesically complete</u> as $t \to 0$, meaning all geodesics approaching $t=0$ extend to arbitrarily values of their affine parameter τ.

However, geodesics may reach $t = \infty$ for finite values of their affine parameter, so the coordinates (x, t) do not describe a geodesically complete spacetime.

On the other hand, the geometry described by (x, t') may be made geodesically complete by extending the coordinate range from $0 < t' < \infty$ to $-\infty < t' < \infty$.

Another example, similar in some respects to the Schwarzschild case, is the Rindler Spacetime,

$$ds^2 = -x^2 dt^2 + dx^2 \,, \quad \begin{cases} -\infty < t < \infty \\ 0 < x < \infty \end{cases}$$

The metric appears singular at $x = 0$. Geodesics terminate with finite affine parameter at $x = 0$, but the curvature is regular as $x \to 0$. Indeed, $R_{\mu\nu\lambda\sigma} = 0$ everywhere in the spacetime.

Null geodesics: $-x^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = 0$

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

$$t = \pm \ln x + \text{const.}$$

$+$ = "outgoing"
$-$ = "ingoing"

Define coordinates
$$u = t - \ln x \,, \quad -\infty < u < \infty$$
$$v = t + \ln x \,, \quad -\infty < v < \infty$$

$v-u = 2\ln x$

$x^2 = e^{v-u}$

$v+u = 2t \longrightarrow t = \frac{v+u}{2}$

$dx = \frac{1}{2} e^{\frac{v-u}{2}} dv - \frac{1}{2} e^{\frac{v-u}{2}} du$

$dt = \frac{1}{2} dv + \frac{1}{2} du$

$ds^2 = -x^2 dt^2 + dx^2$

$= -\frac{e^{v-u}}{4}\left[dv^2 + du^2 + 2\,du\,dv\right] + \frac{e^{v-u}}{4}\left[dv^2 + du^2 - 2\,du\,dv\right]$

$\boxed{ds^2 = -e^{v-u}\,du\,dv}$, $\begin{cases} -\infty < u < \infty \\ -\infty < v < \infty \end{cases}$

Define $U = -e^{-u}$, $V = e^{v}$, $\begin{cases} -\infty < U < 0 \\ 0 < V < \infty \end{cases}$

$\boxed{ds^2 = -dU\,dV}$. Extending the coordinate range to $-\infty < U < \infty$, $-\infty < V < \infty$, the spacetime becomes geodesically complete.

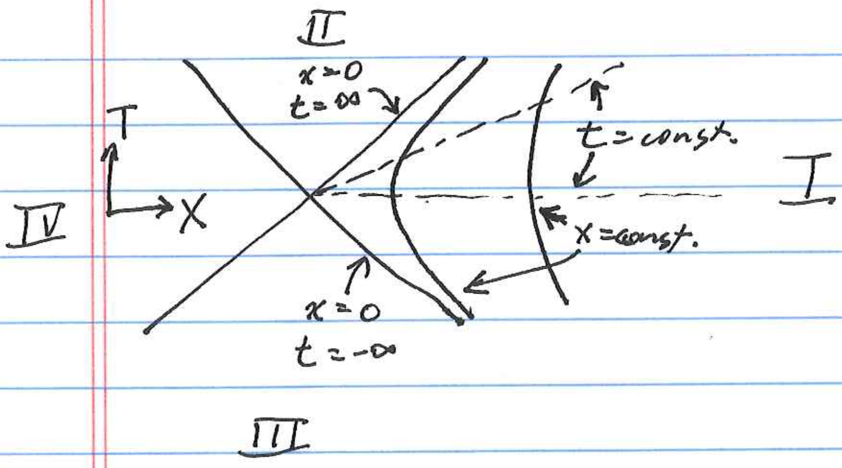
This geometry is just Minkowski space!

Define $T = \frac{U+V}{2}$, $X = \frac{V-U}{2}$

$\boxed{ds^2 = -dT^2 + dX^2}$, $\begin{cases} -\infty < T < \infty \\ -\infty < X < \infty \end{cases}$

In terms of the original coordinates,

$$\boxed{\begin{aligned} x &= (X^2 - T^2)^{1/2} \\ t &= \tanh^{-1}(T/X) \end{aligned}}$$



Rindler spacetime is the region I $(X > |T|)$ of Minkowski spacetime.

Consider a (non geodesic) trajectory $x = \text{const.}$ in the original coordinates.

The proper acceleration is $\boxed{a^\mu = \frac{D}{D\tau}\left(\frac{dx^\mu}{d\tau}\right) = U^\nu D_\nu U^\mu}$
where $U^\mu = \frac{dx^\mu}{d\tau}$,
and $U^\mu U^\nu g_{\mu\nu} = -1 \Longrightarrow U^\mu = \left(\frac{1}{x}, 0\right)$.
The nonvanishing Christoffel symbols are

$$\Gamma^t_{xt} = \Gamma^t_{tx} = -1/x$$

$$\Gamma^x_{tt} = x$$

$$a^\mu = U^\nu\left(\partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda\right)$$
$$= (U^t)^2 \Gamma^\mu_{tt} = \frac{1}{x^2}\Gamma^\mu_{tt}$$

$\boxed{a^x = \frac{1}{x}}$, $a^t = 0$

$\longrightarrow$ The Rindler coordinates (x, t) describe Minkowski space in an accelerated coordinate system.

Kruskal Coordinates

Consider the r, t part of the Schwarzschild metric:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2$$

Null geodesics satisfy

$$-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 0$$

$$\Longrightarrow \left(\frac{dt}{dr}\right)^2 = \left(1-\frac{2GM}{r}\right)^{-2}$$

Solutions: $\boxed{t = \pm r_* + \text{constant},}$

where $\boxed{r_* = r + 2GM \ln(r/2GM - 1)}$

is the "Regge-Wheeler tortoise coordinate."

Note that $\frac{dr_*}{dr} = \left(1-\frac{2GM}{r}\right)^{-1}$.

Define null coordinates u, v:

$$u = t - r_*$$
$$v = t + r_*$$

$\Rightarrow ds^2 = -\left(1-\frac{2GM}{r}\right) du\, dv$, where $r = r(u,v)$ from $r_* = \frac{v-u}{2}$

$$\rightarrow r + 2GM \ln\left(\frac{r}{2GM} - 1\right) = \frac{v-u}{2}$$

$$\frac{r}{2GM} - 1 = e^{(v-u)/4GM}\, e^{-r/2GM}$$

$$= \frac{r}{2GM}\left(1-\frac{2GM}{r}\right)$$

$$\rightarrow \quad ds^2 = -\underbrace{\frac{2GM\, e^{-r/2GM}}{r}}_{\text{Nonsingular as } r\to 2GM,\ (u\to\infty \text{ or } v\to -\infty)} e^{(v-u)/4GM}\, du\, dv$$

Define new coordinates $\quad U = -e^{-u/4GM}$

$$V = e^{v/4GM}$$

$$\Rightarrow \quad ds^2 = -\frac{32(GM)^3 e^{-r/2GM}}{r}\, dU\, dV$$

No singularity at $r = 2GM \quad (U=0 \text{ or } V=0)$

Change coordinates to $T = \frac{U+V}{2}$, $X = \frac{V-U}{2}$, and restore the angular coordinates:

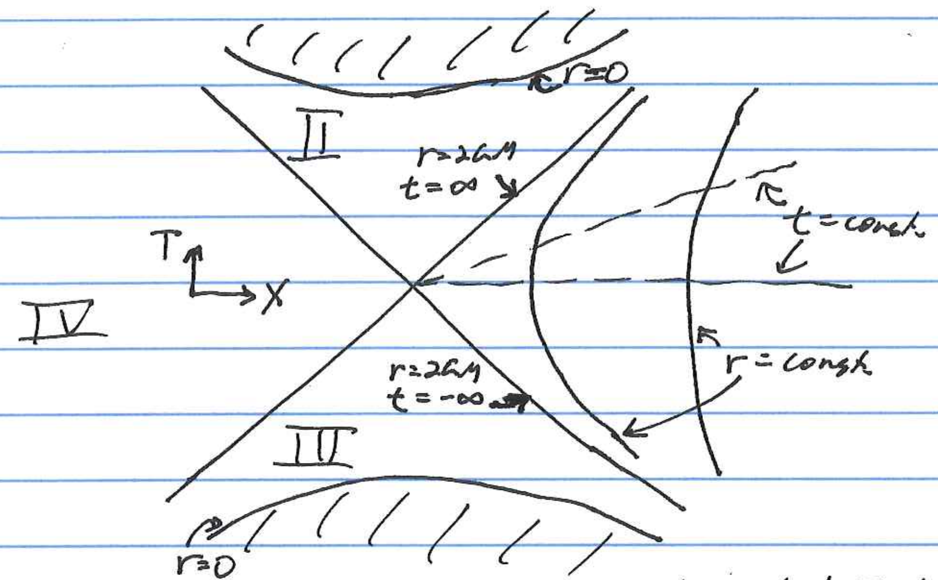
$$\boxed{ds^2 = \frac{32(GM)^3 e^{-r(u,v)/2GM}}{r(u,v)}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

— Schwarzschild metric in Kruskal coordinates

$$\boxed{\begin{aligned} \left(\frac{r}{2GM} - 1\right) e^{r/2GM} &= X^2 - T^2 \\ \frac{t}{2GM} = \ln\left(\frac{T+X}{X-T}\right) &= 2\tanh^{-1}\left(T/X\right) \end{aligned}}$$

The singularity at $r=0$ is a true curvature singularity, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$ as $r \to 0$. The allowed range of coordinates X, T follows from the condition $r > 0$

$$\Rightarrow \quad \boxed{X^2 - T^2 > -1}$$



Kruskal Extension of Schwarzschild Spacetime

Region I: $r > 2GM$ of original Schwarzschild spacetime.

Region II: Black Hole - all observers in this region reach the singularity at $r=0$ in finite proper time.

Region III: White Hole - all observers originated at $r=0$ $(X = -(T^2-1)^{1/2})$ and leave region III in finite proper time.

Region IV: looks like Region I - asymptotically flat, $r > 2GM$.

$t=0$.

Each circle represents a 2-sphere.