General Relativity and Cosmology

This is a course about gravitation, in particular Einstein's general theory of relativity. Einstein developed his theory in the ten years following his "miracle year" (1905) in which he published four incredible papers on the photoelectric effect, Brownian motion, and two papers proposing the Special Theory of Relativity.

The development of general relativity is a wonderful example of how recognition of the appropriate mathematical language to describe a set of physical phenomena leads to a much deeper sense of understanding of those phenomena. Another example of this would be the discovery by Murray Gell-Mann and Yuval Ne'eman that the language of group theory could be used to understand the properties of the zoo of particles discovered in the 1950s and 1960s. (That same group-theoretic language underlies the Standard Model of the strong and electroweak interactions.) In the case of general relativity, the appropriate language is differential geometry, which is used for describing curved spacetime.

From a modern perspective there is an alternative development of the theory, which is the field-theoretic approach taken by Feynman's lectures on gravitation. In that approach we first assume that gravitation is due to some fields which depend on space and time, and then ask what the properties of those fields and interactions must be in order to successfully reproduce experiment and observation.

The field-theoretic approach is not as clever as the geometric approach, but it teaches us some important lessons so we will spend some time developing the subject from this perspective before we discuss the geometric approach.

<u>Newtonian Gravity</u>

We begin with a review of Newton's theory of gravity. In the presence of nongravitational forces $\vec{F}(\vec{x}_a - \vec{x}_b)$ and a gravitational field $\vec{g}$, Newton's second law for particle a takes the form

$$\boxed{m_a \frac{d^2 \vec{x}_a}{dt^2} = m_a \vec{g} + \sum_{b \neq a} \vec{F}(\vec{x}_a - \vec{x}_b).}$$

The gravitational effects can be completely eliminated in a uniform $\vec{g}$, or locally, if $\vec{g}$ varies in space, time:

$$\vec{x}' = \vec{x} - \tfrac{1}{2}\vec{g}t^2, \quad t' = t \;\Rightarrow\; m_a \frac{d^2 \vec{x}_a'}{dt^2} = \sum_{b \neq a} \vec{F}(\vec{x}_a - \vec{x}_b)$$

The coordinate system $(\vec{x}', t')$ describes a freely falling frame, and in that frame there is no evidence of the gravitational field. The equivalence between gravity and acceleration underlies GR.

This is a reflection of the equivalence principle, but we can ask to what extent we know it is valid. Suppose there is a difference between the <u>inertial mass</u> m_i and the <u>gravitational mass</u> m_g.

Galileo knew that objects appear to fall the same way by comparing the motion of different objects rolling down inclined planes, so crudely we can say $m_i \approx m_g$.

Eötvos performed a much more stringent test by comparing the combined forces on hanging plumb bobs due to Earth's gravity (a gravitational effect) and Earth's rotation (an inertial effect). He concluded that for wood (W) and platinum (P),

$$\eta_{(W,P)} \equiv \frac{2\left(\frac{m_{Wg}}{m_{Wi}} - \frac{m_{Pg}}{m_{Pi}}\right)}{\left(\frac{m_{Wg}}{m_{Wi}} + \frac{m_{Pg}}{m_{Pi}}\right)} \lesssim 10^{-9}$$

With a modern torsion balance experiment, the Eöt-Wash experiment found that the analogous observable for Beryllium and titanium is $\eta_{Be,Ti} \lesssim 10^{-13}$. (PRL 100, 041101 (2008))
No evidence for a difference between inertial and gravitational mass has ever been found.

But this is not the end of the story. What is the gravitational field $\vec{g}$ due to a system of particles?

Gravity, being conservative, can be described in terms of a potential ϕ with $\boxed{\vec{g} = -\nabla\phi}$.

The potential satisfies the Poisson equation
$$\boxed{\nabla^2 \phi = 4\pi G\rho},$$
where G is Newton's constant and $\rho(\vec{x}, t)$ is the mass density.

In anticipation of future discussion, consider instead the equation $\nabla^2\phi - m^2\phi = 4\pi G\rho$. (1)
We will analyze the solutions to this equation, and later take $m \to 0$ when we want to describe the gravitational potential.

Fourier transforming $\phi(\vec{x})$, define
$$\tilde{\phi}(\vec{k}) = \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \phi(\vec{x}).$$

Multiplying (1) by $e^{i\vec{k}\cdot\vec{x}}$ and integrating over $\vec{x}$,

integrate by parts twice
$$\int d^3x \, e^{i\vec{k}\cdot\vec{x}} (\nabla^2 - m^2)\phi(\vec{x}) = \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \rho(\vec{x}) \cdot 4\pi G$$
$$= \int d^3x \, (-\vec{k}^2 - m^2) e^{i\vec{k}\cdot\vec{x}} \phi(\vec{x}) \equiv \tilde{\rho}(\vec{k}) \cdot 4\pi G$$
$$= -(\vec{k}^2 + m^2)\tilde{\phi}(\vec{k}).$$

$$\Rightarrow \boxed{\tilde{\phi}(\vec{k}) = -\frac{4\pi G \tilde{\rho}(\vec{k})}{\vec{k}^2 + m^2}}$$

The inverse Fourier transform gives $\phi(\vec{x})$:

$$\phi(\vec{x}) = \int \frac{d^3K}{(2\pi)^3} e^{-i\vec{K}\cdot\vec{x}} \tilde{\phi}(\vec{K})$$

$$= -\int \frac{d^3K}{(2\pi)^3} e^{-i\vec{K}\cdot\vec{x}} \frac{4\pi G\tilde{\rho}(\vec{K})}{\vec{K}^2+m^2}$$

$$= -\int \frac{d^3K}{(2\pi)^3} \frac{e^{-i\vec{K}\cdot\vec{x}}}{\vec{K}^2+m^2} \cdot 4\pi G \int d^3x' e^{i\vec{K}\cdot\vec{x}'} \rho(\vec{x}') \qquad (2)$$

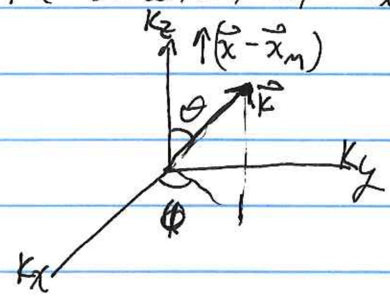
Suppose $\rho(\vec{x}')$ describes a particle of mass M fixed at position $\vec{x}_M$,

$$\rho(\vec{x}') = M\delta^3(\vec{x}'-\vec{x}_M)$$

Then doing the integral over $\vec{x}'$ in (2) gives

$$\phi(\vec{x}) = -\frac{4\pi GM}{8\pi^3} \int d^3K \frac{e^{-i\vec{K}\cdot(\vec{x}-\vec{x}_M)}}{\vec{K}^2+m^2}$$

To do the integral over $\vec{K}$, choose the z-axis to point in the direction of $\vec{x}-\vec{x}_M$, and use spherical coordinates



$$\phi(\vec{x}) = -\frac{4\pi GM}{8\pi^3} \int_0^\infty dK \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, K^2 \sin\theta \frac{e^{-iK|\vec{x}-\vec{x}_M|\cos\theta}}{K^2+m^2}$$

$$= -\frac{4\pi GM}{8\pi^3} \cdot 2\pi \int_0^\infty dK \frac{K^2}{K^2+m^2} \frac{e^{-iK|\vec{x}-\vec{x}_M|} - e^{iK|\vec{x}-\vec{x}_M|}}{-iK|\vec{x}-\vec{x}_M|}$$

$$= +\frac{GM}{\pi} \frac{1}{|\vec{x}-\vec{x}_M|} \int_{-\infty}^{\infty} dK \frac{K^2 e^{+iK|\vec{x}-\vec{x}_M|}}{-iK(K^2+m^2)}$$

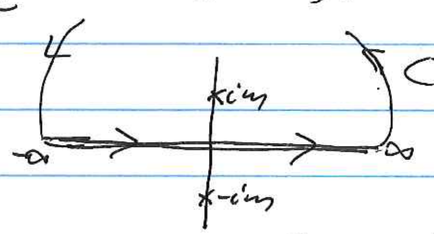
To do the final integral over k, analytically continue the integrand to the complex k-plane and use the residue theorem,

$$\oint dk \, \frac{f(k)}{k-k_0} = 2\pi i \, f(k_0)$$

Closing the contour in the upper-half k-plane (appropriate for the exponential $e^{ik|\vec{x}-\vec{x}'|}$), our integral encircles a pole at $k = im$.

$$\phi(\vec{x}) = \frac{GM}{\pi} \frac{1}{|\vec{x}-\vec{x}_M|} \int_C dk \, \frac{k^2 e^{ik|\vec{x}-\vec{x}_M|}}{-ik(k+im)(k-im)}$$

where the contour is



$$\phi(\vec{x}) = \frac{GM}{\pi} \frac{1}{|\vec{x}-\vec{x}_M|} \cdot \frac{2\pi i \, (im) \exp\left[i(im)|\vec{x}-\vec{x}_M|\right]}{-i\,(2im)}$$

$$\boxed{\phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_M|} e^{-m|\vec{x}-\vec{x}_M|}}$$

With the exponential factor this is known as the Yukawa potential.

With $m \to 0$ this is the usual gravitational potential for a point particle, which is the Coulomb potential (up to an important minus sign)!

$$\boxed{\phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_M|}}$$

With spherical coordinates centered at $\vec{x}_M$, $\phi(\vec{x}) = -\frac{GM}{r}$, and the gravitational field is

$$\vec{g} = -\nabla\phi = -\hat{r}\frac{\partial\phi}{\partial r} = -\frac{GM}{r^2}\hat{r}.$$

In the Newtonian description, gravity is due to a force $\boxed{\vec{F} = m\vec{g} = -\frac{GMm}{r^2}\hat{r}}$ (on an object of mass m).

Incidentally Robert Hooke, Edmund Halley, and Christopher Wren had together conjectured that a $1/r^2$ force law could explain Kepler's observations regarding planetary motion. Hooke became bitter when Newton refused to acknowledge Hooke's influence in the evolution of Newton's thinking on the subject. Christopher Wren attempted to mediate the dispute, but bad feelings between Hooke and Newton persisted.

No violation of the $1/r^2$ force law has been observed down to ~ 20 nm (Eöt-Wash experiment), although at galactic scales and larger we must accept the existence of dark matter to reconcile the motion of stars and galaxies with the $1/r^2$ force law

Inertial Reference Frames (Galileo, Newton)

The coordinate systems in which Newton's laws were presumed to hold were called inertial frames.

Consider the gravitational interaction of a system of point particles:

$$m_a \frac{d^2 \vec{x}_a}{dt^2} = -G \sum_b \frac{m_a m_b (\vec{x}_a - \vec{x}_b)}{|\vec{x}_a - \vec{x}_b|^3} .$$

Consider a new coordinate system $\vec{x}' = R\vec{x} + \vec{v}t + \vec{d}$

$t' = t + t_0$

where R is a rotation, and $\vec{v}$, $\vec{d}$, and t_0 are constants.

3 angles 3 components + 3 + 1 = 10 parameters

This is the 10-parameter family of Galilean transformations.

In the new coordinate system,

$$m_a \frac{d^2}{dt'^2}(R^{-1}\vec{x}') = -G \sum_b \frac{m_a m_b R^{-1}(\vec{x}_a' - \vec{x}_b')}{|\vec{x}_a' - \vec{x}_b'|^3} ,$$

where we used the invariance of the lengths $|\vec{x}_a - \vec{x}_b|$ under rotations.

Acting with R on both sides,

$$m_a \frac{d^2 \vec{x}'}{dt'^2} = -G \sum_b \frac{m_a m_b (\vec{x}_a' - \vec{x}_b')}{|\vec{x}_a' - \vec{x}_b'|^3} .$$

Newton's second law takes the same form in any frame related to a given inertial frame by a Galilean transformation. Invariance of laws of motion under these transformations = Galilean Relativity.

Ernst Mach suggested that the distribution of matter (e.g. the location of the "fixed stars") determined the inertial frames.

To understand Mach's perspective consider this experiment: On a clear night rest your arms at your side and gaze at the stars. They appear fixed. Now spin yourself and notice that simultaneously the stars appear to rotate and your arms are lifted by the centrifugal effect. Why should there be a coincidence between the frames in which the stars are fixed (or moving at uniform velocity) and the frames in which there is no centrifugal effect? One explanation is that the heavens determine the inertial frames.

Einstein's view is similar to Mach's, but with important distinctions. In particular, the inertial frames will depend on the <u>local</u> distribution of matter and not only on some averaged effect of all the matter in the universe.