**Physics 786, Spring 2014**
**Problem Set 2**, Due Wednesday, February 12, 2014.

1. *Gravitational Twin Paradox*

Suppose you and a friend synchronize your perfect watches, and then you move to the 102nd floor of the Empire State Building while your friend stays the ground. How long would you have to wait in order for the time measured on your watches to differ by one second? Whose watch is ahead of the other?

2. *Gravitational Redshift*

Suppose while on the 102nd floor of the Empire State Building you shine a laser towards the ground. By what fraction is the frequency of the laser light increased compared to the emitted frequency when observed on the ground?

3. *Geodesics in Scalar Gravity*

Suppose the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu}\left(1 + 2\Phi(\mathbf{x}, \mathbf{t})\right)$, where $\Phi$ is the gravitational potential. As discussed in class, this would be a good guess for the form of the metric in scalar gravity.

a) If $\Phi = gz$, where $g$ is the acceleration of gravity near the earth and $z$ is the vertical displacement from the ground, what are the nonvanishing components of the Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$?

b) In the Newtonian approximation, use the geodesic equation to calculate the acceleration of a freely falling particle $\frac{d^2\mathbf{x}}{dt^2}$.

c) Along a lightlike trajectory $x^{\mu}(t)$, the geodesics satisfy

$$g_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} = 0.$$

Show that the lightlike trajectories in any potential $\Phi$ are straight lines, as in special relativity. The absence of bending of light in scalar gravity is in contradiction to observation.

4. *The Angular Momentum Tensor*

Define

$$M^{\gamma\alpha\beta} \equiv x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma},$$

where $T^{\beta\gamma}$ is the symmetric conserved energy-momentum tensor.

a) Show that $M^{\gamma\alpha\beta}$ is conserved, *i.e.*

$$\partial_\gamma M^{\gamma\alpha\beta} = 0.$$

b) What corresponding rank-2 tensor is time independent as a result of this conservation law?

c) Explain how the spatial components of the tensor found in part (b) are related to the angular momentum of the system.

5. *Geodesics on the 2-sphere*

In spherical coordinates, the length element on the 2-sphere of radius $R$ takes the form

$$ds^2 = R^2 \left( d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

a) With $x^1 = \theta$ and $x^2 = \phi$, the metric $g_{ij} = g_{ji}$ is defined such that $ds^2 = g_{ij} dx^i dx^j$, summed over $i$ and $j$. What are the components of $g_{ij}$, written as a 2×2 matrix?

b) Find the nonvanishing components of the connection

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right).$$

c) Consider a path parametrized by a parameter $t$. The paths of shortest distance satisfy the geodesic equation:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0.$$

Show that arcs along the equator $\theta = \pi/2$ are geodesics on the 2-sphere.