Physics 786, Spring 2014
Problem Set 1 Due Wednesday, February 5, 2014.

1. *Lorentz tensors*

a) If $T^{\mu\nu}$ and $B_{\mu\nu}$ are tensors under Lorentz transformations, prove that $T^{\mu}_{\ \nu}B_{\mu}^{\ \nu}$ is a Lorentz scalar.

b) If $A^{\mu}(x)$ is a vector field, show that $\partial_{\mu}A_{\nu}(x)$ transforms like a (0,2) tensor under Lorentz transformations.

c) Write down all Lorentz invariants that contain two factors of either $A^{\mu}(x)$ or its first derivatives.

d) If $h_{\mu\nu}(x)$ is a tensor field, write down all Lorentz invariants that contain two factors of either $h_{\mu\nu}(x)$ or its first derivatives.

e) Assume that the Minkowski metric, $\eta_{\mu\nu}$, transforms as a (0,2) tensor under Lorentz transformations. From the defining property of the Lorentz transformations, show that $\eta_{\mu\nu}$ is Lorentz invariant.

2. *The Levi-Civita tensor*

The Levi-Civita tensor $\epsilon^{\mu\nu\lambda\sigma}$ is antisymmetric under exchange of any two of its indices, with $\epsilon^{0123} = +1$. Show that $\epsilon^{\mu\nu\lambda\sigma}$ is invariant under Lorentz transformations with $\det\Lambda = +1$.

Note that the determinant of a 4×4 matrix A with components $A_{\mu\nu}$, where $\mu, \nu \in \{0, 1, 2, 3\}$, can be written

$$\det A = \sum_{\mu\nu\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} A_{0\mu}A_{1\nu}A_{2\lambda}A_{3\sigma}.$$

3. *Lorentz-covariant form of (some of) Maxwell's equations*

Maxwell's equations can be written in a Lorentz-covariant form in terms of the antisymmetric field-strength tensor $F^{\mu\nu}$. The components of $F^{\mu\nu}$ are:

$$\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix},$$

where E_i and B_i are the components of the electric and magnetic field, respectively.

a) What are the components of $F_{\mu\nu}$?

b) Write the equation

$$\partial_\mu F^{\mu\nu} = 0$$

in terms of the $\mathbf{E}$ and $\mathbf{B}$. Consider separately the components $\nu = 0$ and $\nu = i \in \{1, 2, 3\}$. Compare with Maxwell's equations in terms of the electric and magnetic fields.

4. *Lorentz transformation of the electromagnetic field*

Suppose $\mathbf{B}$=0 in some reference frame. Consider a Lorentz boost by speed v in the z-direction. By considering the Lorentz transformation of $F^{\mu\nu}$ determine the components of the electric field $\mathbf{E}'$ and magnetic field $\mathbf{B}'$ in the boosted frame in terms of v and the electric field $\mathbf{E}$ in the original frame.

5. *Free fall*

Complete the derivation from class that along the trajectory of a freely falling particle,

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0.$$