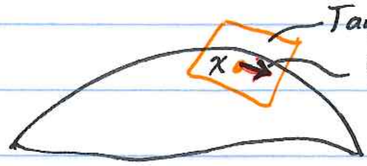
Zee 1.7

Covariant Derivative

Vector fields on a manifold live in the tangent space, i.e. in the tangent plane at each pt. in the manifold.



Tangent plane at x

$$\vec{W}(x) = W^\mu(x)\, \vec{e}_\mu(x)$$

This discussion is the same in space or spacetime, so we use Greek indices for generality.

Ordinary derivatives of $\vec{W}(x)$ do not generally remain in the tangent space:

$$\partial_\nu \vec{W}(x) = \partial_\nu (W^\mu \vec{e}_\mu) = (\partial_\nu W^\mu)\, \vec{e}_\mu + W^\mu \partial_\nu \vec{e}_\mu ,$$

or, using $\partial_\nu \vec{e}_\mu = \Gamma^\lambda_{\mu\nu}\, \vec{e}_\lambda + K_{\mu\nu}\, \vec{n}$,

$$\begin{aligned}\partial_\nu \vec{W} &= (\partial_\nu W^\mu)\, \vec{e}_\mu + W^\lambda \Gamma^\mu_{\lambda\nu}\, \vec{e}_\mu + W^\mu K_{\mu\nu}\, \vec{n} \\ &= (\partial_\nu W^\mu + \Gamma^\mu_{\lambda\nu} W^\lambda)\, \vec{e}_\mu + W^\mu K_{\mu\nu}\, \vec{n}\end{aligned}$$

The projection of $\partial_\nu \vec{W}$ onto the tangent plane defines the covariant derivative of $\vec{W}(x)$:

$$\boxed{D_\nu \vec{W} \equiv (\partial_\nu W^\mu + \Gamma^\mu_{\lambda\nu} W^\lambda)\, \vec{e}_\mu \equiv (D_\nu W^\mu)\, \vec{e}_\mu}$$

Note that $D_\nu W^\mu$ depends in general on all components of W^α, not just the component $\alpha = \mu$.

Zee V.6

The covariant derivative $D_\nu \vec{W}$ lives in the tangent space, and $D_\nu W^\mu$ transforms as a tensor. This is another way to introduce the covariant derivative: we ask for a derivative that transforms covariantly under coordinate transformations.

The Affine Connection is not a tensor under general coordinate transformations.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$, $\xi^{\alpha}(x)$ is a locally inertial coordinate system.

In the coordinate system x',

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

chain rule $\rightarrow$
$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right)$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right]$$

$$\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

$$\boxed{\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}}$$

Differentiation of a tensor does not generally yield another tensor.

Under the transformation $x \to x'$, $V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$ for vector V^{μ}.

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}}}_{\text{Tensor-like transformation}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}}_{\text{non-tensorlike.}}$$

semicolon, not the letter ;

However, the combination $D_\lambda V^\mu \equiv V^\mu{}_{;\lambda} = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} V^\kappa$,

the covariant derivative, is a tensor:

$$V'^\mu{}_{;\lambda} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial x'^\lambda} V^\nu{}_{;\rho}$$

To show this we rewrite the transformation of $\Gamma^\lambda_{\mu\nu}$ in a different way.

Use $\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu} = \delta^\lambda_\nu$

$$\frac{\partial}{\partial x'^\mu} : \quad \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma} = 0$$

$$\Rightarrow \boxed{\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}}$$

It is now straightforward to show that the non-tensorlike term in the transformation of $\Gamma^\lambda_{\mu\nu}$ cancels the non-tensorlike term in the transformation of $\partial_\lambda V^\mu$ in the combination $\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\kappa} V^\kappa$. (Exercise)

Similarly, the covariant derivative of a covariant vector is

$$\boxed{D_\nu V_\mu \equiv V_{\mu;\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda}$$

Under a coordinate transformation $D_\nu V_\mu$ transforms as a tensor:

$$V'_{\mu;\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} V_{\rho;\sigma} \qquad \text{(Exercise)}$$

In general, covariant derivatives of tensors involve a sum of terms, each involving one factor of $\Gamma^{\lambda}_{\mu\nu}$, one for each index on the tensor.

<u>Example</u>: $$T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\sigma}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}{}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}{}_{\lambda} - \Gamma^{\kappa}_{\lambda\rho} T^{\mu\sigma}{}_{\kappa}$$

<u>Exercise</u>: Check that $T^{\mu\sigma}{}_{\lambda;\rho}$ is a tensor.

<u>Properties of Covariant Derivatives</u>

1) $(\alpha A^{\mu}{}_{\nu} + \beta B^{\mu}{}_{\nu})_{;\lambda} = \alpha A^{\mu}{}_{\nu;\lambda} + \beta B^{\mu}{}_{\nu;\lambda}$ linearity

2) $(A^{\mu}{}_{\nu} B^{\lambda})_{;\rho} = A^{\mu}{}_{\nu;\rho} B^{\lambda} + A^{\mu}{}_{\nu} B^{\lambda}{}_{;\rho}$ Leibniz rule

3) $T^{\mu\lambda}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\lambda}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\lambda}{}_{\lambda}$ Derivative of contraction works as if contracted indices were not there

<u>Covariant Differentiation of the Metric</u>

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}$$

$$= 0$$ (using definition of $\Gamma^{\rho}_{\lambda\mu}$ in terms of $g_{\mu\nu}$)

We can also show this by considering a locally inertial coordinate system, in which $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = 0$, $\Gamma^{\rho}_{\lambda\mu} = 0$ at some pt P. But $g_{\mu\nu;\lambda}$ is a tensor, so in a general coordinate system, $g_{\mu\nu;\lambda}$ remains zero.

Similarly, $g^{\mu\nu}{}_{;\lambda} = 0$

$$\delta^{\mu}{}_{\nu;\lambda} = 0$$

* Covariant Differentiation Commutes w/ Raising, Lowering Indices

$$(g^{\mu\nu}V_\nu)_{;\lambda} = \cancel{g^{\mu\nu}{}_{;\lambda}}^{\,0}\, V_\nu + g^{\mu\nu}V_{\nu;\lambda}$$

$$= g^{\mu\nu}V_{\nu;\lambda}$$

Special Cases of Covariant Differentiation

Covariant Derivative of a Scalar S: $\boxed{S_{;\mu} = \frac{\partial S}{\partial x^\mu}}$

Covariant Curl: Recall $V_{\mu;\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda$

symmetric in $\mu \leftrightarrow \nu$.

curl: $\boxed{V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}}$ = ordinary curl.

The covariant divergence of a covariant vector can be written in terms of $g = \det(g_{\mu\nu})$ using the following identity:

$$\boxed{\mathrm{Tr}\left\{M^{-1}(x)\,\frac{\partial}{\partial x^\lambda}M(x)\right\} = \frac{\partial}{\partial x^\lambda}\ln\det M(x)}$$

Proof: If $x^\lambda \to x^\lambda + \delta x^\lambda$, then

$$\delta \ln\det M = \ln\det(M + \delta M) - \ln\det M$$

$$\delta \ln \det M = \ln\left(\frac{\det(M+\delta M)}{\det M}\right)$$

$$= \ln \det\left(M^{-1}(M+\delta M)\right)$$

$$= \ln \det\left(\mathbb{1} + M^{-1}\delta M\right)$$

$$\approx \ln\left(1 + \operatorname{Tr} M^{-1}\delta M\right)$$

$$\approx \operatorname{Tr} M^{-1}\delta M$$

$$\lim_{\delta x^\lambda \to 0} \frac{\delta \ln \det M}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} \ln \det M$$

$$= \operatorname{Tr}\left(M^{-1}\frac{\partial M}{\partial x^\lambda}\right) \qquad \blacksquare$$

with $M = g_{\rho\mu}$, $g^{\mu\rho}\frac{\partial}{\partial x^\lambda} g_{\rho\mu} = \frac{\partial}{\partial x^\lambda}\ln g$

$$= \frac{2}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}\sqrt{g}$$

Then $\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\left\{\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\mu\lambda}\right\}$

$$= \frac{1}{2} g^{\mu\rho}\,\partial_\lambda g_{\rho\mu}$$

$$\boxed{\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{g}}\partial_\lambda\sqrt{g}}$$

$$\Rightarrow \boxed{V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}\,V^\mu}$$

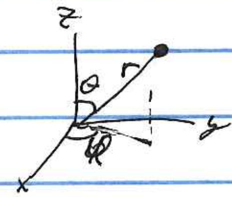
Covariant Laplacian / D'Alembertian

If $\phi(x)$ is a scalar,
$$\phi_{;\mu}{}^{;\mu} = \left(g^{\mu\nu}\phi_{;\nu}\right)_{;\mu} = \left(g^{\mu\nu}\partial_\nu\phi\right)_{;\mu}$$

$$\boxed{\phi_{;\mu}{}^{;\mu} = \frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi\right)}$$

In flat space these formulae allow us to compute the gradient, divergence, and curl in arbitrary coordinates.

Example: Laplacian in spherical coordinates



$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/(r^2\sin^2\theta) \end{pmatrix}$$

$$g \equiv \det g_{\mu\nu} = r^4\sin^2\theta$$

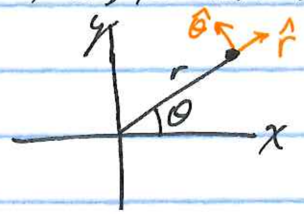
$$\nabla^2\phi = \frac{1}{\sqrt{g}}\,\partial_i\left(\sqrt{g}\,g^{ij}\,\partial_j\phi\right)$$

$$= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(r^2\sin\theta\cdot\frac{1}{r^2}\,\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(r^2\sin\theta\cdot\frac{1}{r^2\sin^2\theta}\,\frac{\partial\phi}{\partial\varphi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2\phi}{\partial\varphi^2}$$

Example: Divergence in 2D Polar coordinates.

$\partial_\mu V^\mu$ is not a scalar under general coordinate transformations. $D_\mu V^\mu$ is a scalar.



Polar coordinates:

$$x = r\cos\theta \iff r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$. Let (x,y) be the unprimed coords, (r,θ) the primed coords.

$$V^r = \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y = \cos\theta\, V^x + \sin\theta\, V^y$$

$$V^\theta = \frac{\partial \theta}{\partial x} V^x + \frac{\partial \theta}{\partial y} V^y = -\frac{\sin\theta}{r} V^x + \frac{\cos\theta}{r} V^y$$

$$D_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, V^\mu\right)$$

$$ds^2 = dr^2 + r^2 d\theta^2 \rightarrow g = r^2 \quad (\text{determinant of } g_{\mu\nu})$$

$$D_\mu V^\mu = \frac{1}{r}\left[\frac{\partial}{\partial r}(r V^r) + r \frac{\partial}{\partial \theta} V^\theta\right]$$

$$= \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{\partial}{\partial \theta} V^\theta}_{\nabla\cdot\vec{V} \text{ in polar coordinates.}}$$

Sometimes V^θ is rescaled by r so that the dimensions of V^θ and V^r are the same. If $\tilde{V}^\theta \equiv r V^\theta$, then

$$\boxed{D_\mu V^\mu = \frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{1}{r}\frac{\partial}{\partial \theta}\tilde{V}^\theta}$$