

lec 1.5-1.7

## Scalars, Vectors and Tensors under General Coord. Transfs.

Contravariant Vectors transform like  $dx^\mu$ :

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$$

Covariant Vectors transform like:

$$U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} U_{\nu}$$

Tensors transform with additional factors of  $\frac{\partial x}{\partial x'}$  or  $\frac{\partial x'}{\partial x}$ :

$$T'^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x^{\delta}} T^{\alpha}_{\beta\gamma}$$

Scalars are invariant:  $\phi' = \phi$

Metric: 
$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

Inverse metric: 
$$g^{\lambda\mu}, \quad g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu}$$

If 
$$g'^{\lambda\mu} = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} g^{\alpha\beta} \quad (2,0) \text{ tensor, then}$$

$$g'^{\lambda\mu} g'_{\mu\nu} = \left( \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} g^{\alpha\beta} \right) \left( \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} g_{\gamma\delta} \right)$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \underbrace{\delta^{\gamma}_{\beta} g^{\alpha\beta} g_{\gamma\delta}}_{g^{\alpha\delta} g_{\delta\gamma} = \delta^{\alpha}_{\gamma}} \quad \text{using } \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} = \delta^{\beta}_{\beta}$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}$$

$\Rightarrow g'^{\lambda\mu} g'_{\mu\nu} = \delta^{\lambda}_{\nu}$ ,  $g'^{\lambda\mu}$  is the inverse of  $g'_{\mu\nu}$ .

Tensors can be added, multiplied, and indices lowered and raised with the metric or inverse metric (respectively), yielding new tensors.

Examples:

$$C^M = A^M + B^M \quad \text{vector}$$

$$D^{MN} = A^M B^N \quad \text{tensor}$$

$$E^M = g^{MN} E_N \quad \text{contravariant vector}$$

$$E_M = g_{MN} E^N \quad \text{covariant vector}$$

$$F = \underbrace{A^M B_M}_{\text{contracted indices}} \equiv A^M B^N g_{MN} = A_M B^M g^{MN} \quad \text{Scalar}$$

Exercise: Show that the tensors in these examples transform as claimed.

Note that indices are now contracted with  $g_{MN}$ , not  $g^{MN}$ .

Wessberg  
4.4-4.8,  
sect. 5

Tensor Densities - transform with extra factors  
of  $\det\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)$  or  $\det\left(\frac{\partial x'^\rho}{\partial x^\mu}\right)$  under  
coordinate transformations  $x \rightarrow x'$ .

Example:  $g \equiv |\det g_{\mu\nu}|$  (The notation is different than  
for any other tensor.  $T \equiv T^{\mu}_{\nu}$  is  
a trace. For the metric  $g^{\mu}_{\mu} = 4$ .)  
 $\uparrow$  metric

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$$

$$\det g'_{\mu\nu} = \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^2 \det g_{\rho\sigma}$$

$$= \frac{1}{\det\left(\frac{\partial x'}{\partial x}\right)^2} \det(g_{\rho\sigma}) \rightarrow \text{tensor density of weight } -2$$

$\det\left(\frac{\partial x'}{\partial x}\right)$  is the Jacobian of the transformation  
 $x \rightarrow x'$

The usual volume measure  $d^4x$  is a scalar density of  
weight 1:

$$d^4x' = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| d^4x$$

The measure  $\boxed{\sqrt{g} d^4x}$  is a scalar. Integrals of

scalars against this measure will be coordinate invariant.

(This also gives the volume element in an arbitrary coordinate system.)

Example:  $\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$

$\uparrow$   
Levi-Civita tensor (density)

tensor density of weight  $-1$ .

Example: In 3D Euclidean space, the volume element in Cartesian coordinates is  $d^3x' = dx dy dz$



In spherical coordinates,  $x = r \sin\theta \cos\varphi$   
 $y = r \sin\theta \sin\varphi$   
 $z = r \cos\theta$

Define  $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \varphi$   
 $x'^1 = x$ ,  $x'^2 = y$ ,  $x'^3 = z$

$$\frac{\partial x'}{\partial x} = \begin{pmatrix} \sin\theta \cos\varphi & r \cos\theta \cos\varphi & -r \sin\theta \sin\varphi \\ \sin\theta \sin\varphi & r \cos\theta \sin\varphi & r \sin\theta \cos\varphi \\ \cos\theta & -r \sin\theta & 0 \end{pmatrix}$$

$$\det\left(\frac{\partial x'}{\partial x}\right) = r^2 \sin\theta \quad (\text{Exercise})$$

$$\Rightarrow d^3x' = r^2 \sin\theta dr d\theta d\varphi$$

$$ds^2 = dx^2 + dy^2 + dz^2 \rightarrow g'_{ij} = \delta_{ij}, \quad \det(g') = 1$$

$$\sqrt{g'} d^3x' = dx dy dz$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \Rightarrow g_{ij} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

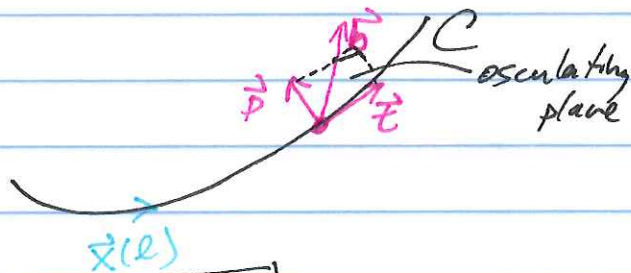
$$\det(g) = r^4 \sin^2\theta, \quad \sqrt{g} = r^2 \sin\theta$$

$$\sqrt{g} d^3x = r^2 \sin\theta dr d\theta d\varphi$$

Ex 1.7

## Classical Differential Geometry

Consider a curve  $C$  given by  $\vec{x}(l)$  parametrized by the arc length  $l$  along the curve.



$$\boxed{\vec{T} \equiv \dot{\vec{x}} = \frac{d\vec{x}}{dl}} = \text{unit tangent vector}$$

$$dl^2 = d\vec{x} \cdot d\vec{x} \Rightarrow \vec{T} \cdot \vec{T} = 1 \quad (\text{unit vector})$$

$$\frac{d}{dl}(\vec{T} \cdot \vec{T}) = 2\vec{T} \cdot \dot{\vec{T}} = 0$$

Unit vector  $\vec{P}(l)$ , such that  $\boxed{\dot{\vec{T}} = \kappa \vec{P}}$

$\vec{P} \equiv$  principal normal vector,  $\boxed{\vec{P} \cdot \vec{T} = 0}$

$$\boxed{\kappa = |\dot{\vec{T}}|}$$

$$\boxed{\vec{b}(l) \equiv \vec{T}(l) \times \vec{P}(l)} \quad \text{binormal vector, } \boxed{\vec{b} \cdot \vec{b} = 1}$$

$$\vec{T} \cdot \vec{b} = \vec{T} \cdot (\vec{T} \times \vec{P}) = 0 \Rightarrow \frac{d}{dl}(\vec{T} \cdot \vec{b}) = \dot{\vec{T}} \cdot \vec{b} + \vec{T} \cdot \dot{\vec{b}} = 0$$

$$\dot{\vec{T}} \cdot (\vec{T} \times \vec{P}) + \vec{T} \cdot \dot{\vec{b}} = 0. \quad \text{Using } \dot{\vec{T}} = \kappa \vec{P}, \quad \vec{T} \cdot (\vec{T} \times \vec{P}) = 0$$

$$\Rightarrow \boxed{\vec{T} \cdot \dot{\vec{b}} = 0}$$

$$\frac{d}{dl}(\vec{b} \cdot \vec{b}) = 2\vec{b} \cdot \dot{\vec{b}} = 0 \Rightarrow \boxed{\vec{b} \cdot \dot{\vec{b}} = 0}$$

$$\left. \begin{array}{l} \vec{T} \cdot \dot{\vec{b}} = 0 \\ \vec{b} \cdot \dot{\vec{b}} = 0 \end{array} \right\} \boxed{\dot{\vec{b}} = -\tau \vec{P}}$$

$$|\tau| = |\dot{\vec{b}}|$$

$$\dot{\rho} = \dot{\mathbf{b}} \times \dot{\mathbf{t}}$$

$$\dot{\rho} = \dot{\mathbf{b}} \times \dot{\mathbf{t}} + \dot{\mathbf{b}} \times \dot{\mathbf{t}} = -\tau \dot{\rho} \times \dot{\mathbf{t}} + \dot{\mathbf{b}} \times \kappa \dot{\rho}$$

$$\boxed{\dot{\rho} = \tau \dot{\mathbf{b}} - \kappa \dot{\mathbf{t}}}$$

Hence, we have found 
$$\left. \begin{aligned} \dot{\mathbf{t}} &= \kappa \dot{\rho} \\ \dot{\mathbf{b}} &= -\tau \dot{\rho} \\ \dot{\rho} &= \tau \dot{\mathbf{b}} - \kappa \dot{\mathbf{t}} \end{aligned} \right\} \text{Frenet-Serret equations.}$$

Define  $\Psi \equiv \begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\rho} \\ \dot{\mathbf{b}} \end{pmatrix}$ . Then,

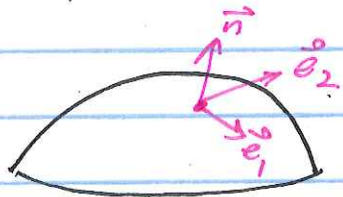
$$\dot{\Psi} = A \Psi, \text{ with } A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

## Surfaces

Consider a surface in 3-Dimensional Euclidean space, parametrized by two coordinates,  $x^i$ ,  $i=1,2$ :  $\vec{X}(x^1, x^2)$  specifies the pts on the surface.

Define the two vectors  $\boxed{\vec{e}_i \equiv \partial_i \vec{X} = \frac{\partial \vec{X}}{\partial x^i}}$  — basis vectors on the surface.

Tangent plane: set of points  $u^i \vec{e}_i$ , for real  $u^i$ .



$$\vec{n} = \frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|} \text{ unit normal to surface.}$$

Distances on the surface:

$$d\vec{X} = \partial_i \vec{X} dx^i$$

$$dS^2 = d\vec{X} \cdot d\vec{X} = (\partial_i \vec{X} \cdot \partial_j \vec{X}) dx^i dx^j$$

$$= \underbrace{\vec{e}_i \cdot \vec{e}_j}_{g_{ij}} dx^i dx^j$$

$g_{ij} = \vec{e}_i \cdot \vec{e}_j$

Induced Metric  
(induced by the embedding in 3D Euclidean space)

$$\partial_j \vec{e}_i = \tilde{\Gamma}_{ij}^k \vec{e}_k + K_{ij} \vec{n} \quad \text{for some } \tilde{\Gamma}_{ij}^k, K_{ij}.$$

$$\vec{n} \cdot \partial_j \vec{e}_i = \tilde{\Gamma}_{ij}^k \vec{n} \cdot \vec{e}_k + K_{ij} \vec{n} \cdot \vec{n}$$

$$\Rightarrow \boxed{K_{ij} = \vec{n} \cdot \partial_j \vec{e}_i} \quad \text{Gauss's equation}$$

$$\vec{e}_l \cdot \partial_j \vec{e}_i = \tilde{\Gamma}_{ij}^k \vec{e}_l \cdot \vec{e}_k + K_{ij} \vec{e}_l \cdot \vec{n}$$

$$= \tilde{\Gamma}_{ij}^k g_{lk}$$

Because  $\partial_j \vec{e}_i = \partial_j \partial_i \vec{X} = \partial_i \partial_j \vec{X} = \partial_i \vec{e}_j$ ,  
 $\tilde{\Gamma}_{ij}^k$  and  $K_{ij}$  are symmetric in  $i, j$ .

$$\tilde{\Gamma}_{ij}^k g_{re} g^{em} = \boxed{\tilde{\Gamma}_{ij}^m = g^{lm} \vec{e}_l \cdot \partial_j \vec{e}_i}$$

$$\text{Using } \partial_j g_{li} = \vec{e}_i \cdot \partial_j \vec{e}_l + \vec{e}_l \cdot \partial_j \vec{e}_i,$$

$$\boxed{\tilde{\Gamma}_{ij}^m = \frac{1}{2} g^{lm} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})}$$

Exercise.

Hence,  $\tilde{\Gamma}_{ij}^m = \Gamma_{ij}^m$ , the Christoffel symbols.

The  $\Gamma_{ij}^k$  tell us how the tangent plane rotates about the normal vector on the surface, while  $K_{ij}$  tells us how the tangent plane moves in the embedding space.