The Gravitational Coupling λ

In the harmonic gauge, the linearized Einstein equations took the form,

$$\partial_\alpha \partial^\alpha h_{\mu\nu} = -\lambda \left(T_{\mu\nu} - \tfrac{1}{2} \eta_{\mu\nu} T \right),$$

but we still need to identify the constant λ, which determines the strength of the gravitational coupling to $T_{\mu\nu}$.

In the Newtonian limit, $T^{00} = T_{00} = \rho$, $T^\mu{}_\mu \propto T^0{}_0 = -\rho$,

$$\partial_\alpha \partial^\alpha h_{00} \approx -\lambda \left(\rho - \tfrac{1}{2} (-1)(-\rho) \right) = -\tfrac{\lambda}{2} \rho$$

Earlier, by comparing the geodesic equation for a freely falling particle in the Newtonian limit to $\vec{a} = -\nabla \phi$, we deduced $h_{00} = -2\phi$, and $\nabla^2 \phi = 4\pi G \rho$.

For static h_{00}, we have

$$\nabla^2 h_{00} = -2\nabla^2 \phi = -8\pi G \rho \left. \right\} \quad \boxed{\lambda = 16\pi G}$$
$$= -\tfrac{\lambda}{2} \rho \left. \right\}$$

With this identification of the coupling λ, we now have completely determined the linearized Einstein equations, which in arbitrary gauge are,

$$\boxed{\partial_\alpha \partial^\alpha h_{\mu\nu} - \left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} \right) + \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\alpha \partial^\alpha h = -16\pi G \, T_{\mu\nu}}$$

Conservation of $T_{\mu\nu}$?

Conservation of $T_{\mu\nu}$ played an important role in our determination of the equation of motion for the gravitational field $h_{\mu\nu}$. A consequence of $\partial_\mu T^{\mu\nu} = 0$, is time-independence of the energy of the system described by $T^{\mu\nu}$. However, gravitational radiation carries off energy (it can do work), so either the total energy of gravity + matter is not conserved, or the matter energy alone is not conserved. To examine this issue we will consider the $T_{\mu\nu}$ of a gravitating particle.

To determine $T_{\mu\nu}$, we consider the relativistic relations for the energy and momentum of a particle:

Energy $\quad E = mc^2 \gamma = mc^2 \dfrac{dt}{d\tau}$, where $d\tau^2 = dt^2 - \dfrac{1}{c^2} d\vec{x}^2$

$$E = \int d^3x \, \underbrace{mc^2 \frac{dt}{d\tau} \delta^3(\vec{x} - \vec{x}(t))}_{c^2 T^{00} = \text{Energy density}}$$

Momentum $\quad p^i = m \dfrac{dx^i}{dt} \gamma = m \dfrac{dx^i}{dt} \dfrac{dt}{d\tau} = m \dfrac{dx^i}{d\tau}$

$$= \int d^3x \, \underbrace{m \frac{dx^i}{d\tau} \delta^3(\vec{x} - \vec{x}(t))}_{T^{0i}}$$

$$T^{00} = m \frac{dt}{d\tau} \delta^3(\vec{x} - \vec{x}(t)) = m \frac{dt}{d\tau} \frac{dt}{d\tau} \frac{d\tau}{dt} \delta^3(\vec{x} - \vec{x}(t))$$

$$T^{0i} = m \frac{dx^i}{d\tau} \delta^3(\vec{x} - \vec{x}(t)) = m \frac{dt}{d\tau} \frac{dx^i}{d\tau} \frac{d\tau}{dt} \delta^3(\vec{x} - \vec{x}(t))$$

A Lorentz-covariant form of $T^{\mu\nu}$ consistent with these T^{00} and T^{0i} is

$$T^{\mu\nu} = m \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \delta^3(\vec{x} - \vec{x}(t)) \frac{d\tau}{dt}.$$

This is the energy-momentum tensor, or stress-energy tensor, of a particle of mass m moving along a trajectory $x^{\mu}(\tau)$. To make the covariance of $T^{\mu\nu}$ explicit we can write

$$T^{\mu\nu} = m \int dt \, \delta^4(x - x(t)) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \frac{d\tau}{dt}$$

$$T^{\mu\nu} = m \int d\tau \, \delta^4(x - x(\tau)) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}$$

Now consider conservation of this $T^{\mu\nu}$:

$$\partial_{\mu} T^{\mu\nu} = m \int d\tau \, \underbrace{\partial_{\mu} \delta^4(x - x(\tau)) \frac{dx^{\mu}}{d\tau}}_{\frac{d}{d\tau} \delta^4(x - x(\tau))} \frac{dx^{\nu}}{d\tau}$$

$$\overset{\text{by parts}}{=} m \int d\tau \, \delta^4(x - x(\tau)) \frac{d^2 x^{\nu}}{d\tau^2}$$

$$= m \int d\tau \, \delta^4(x - x(\tau)) \left(- \Gamma^{\nu}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} \right)$$

by the eq. of motion for the particle (the geodesic eq.)

$$= - \Gamma^{\nu}_{\alpha\beta} T^{\alpha\beta}$$

$$\Rightarrow \boxed{\partial_{\mu} T^{\mu\nu} = - \Gamma^{\nu}_{\alpha\beta} T^{\alpha\beta}}$$

In our linear theory, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,
$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$$

so that $g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha + O(h^2)$. (Recall that $g^{\mu\nu}$ with upper indices is defined as the inverse (as a matrix) of $g_{\mu\nu}$ with lower indices.)

(✱) $\partial_\mu T^{\mu\nu} = -\Gamma^\nu_{\alpha\beta} T^{\alpha\beta}$
$$= -\frac{1}{2} \eta^{\nu\rho} \left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta} \right) T^{\alpha\beta} + O(h^2)$$
$$\neq 0$$

Evidently, $T^{\mu\nu}$ of matter alone is not conserved, but we can attempt to find a gravitational contribution to the stress-energy tensor such that the sum of the matter + gravity contributions is **conserved**, at least to lowest order in $h_{\mu\nu}$.

We assume that the equation (✱) is valid more generally, for example for a collection of particles.

We also assume that to lowest order in $h_{\mu\nu}$, the linearized Einstein equations are satisfied. This allows us to replace $T^{\alpha\beta}$ on the right-hand side of (✱) by derivatives of $h_{\mu\nu}$.

We can then find tensors $\chi^{\mu\nu}$ bilinear in h and its derivatives such that
$$\partial_\mu \left(T^{\mu\nu} + \chi^{\mu\nu} \right) = 0 \quad , \text{ i.e.}$$

$$\boxed{ \partial_\mu \chi^{\mu\nu} = +\frac{1}{2} \eta^{\nu\rho} \left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta} \right) T^{\alpha\beta} + O(h^2) }$$

The most general expression for $\chi^{\mu\nu}$ includes
many terms,

$$\chi^{\mu\nu} = a \, \partial_\gamma h^{\mu\nu} \, \partial_\beta h^{\gamma\beta} + b \, \partial_\gamma h^{\mu\beta} \, \partial_\beta h^{\nu\gamma} + \dots$$

Suppose we have found a $\chi^{\mu\nu}$ that satisfies $\partial_\mu(T^{\mu\nu} + \chi^{\mu\nu}) = 0$
to $O(h \cdot T)$.

But now the linearized eq. of motion for $h_{\mu\nu}$ cannot
be exactly satisfied, because $\partial_\mu T^{\mu\nu} = 0$ is a
consequence of that equation.

So we ask for a new, nonlinear equation for $h_{\mu\nu}$
with $T^{\mu\nu}$ as its source.

The form of $\chi^{\mu\nu}$ is not completely determined by
knowledge of $\partial_\mu \chi^{\mu\nu}$, so we can impose a further
constraint that a local invariance like $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ $+ O(\xi h)$
leave the equation for $h_{\mu\nu}$ invariant.

Alternatively, we can insist that the equation for $h_{\mu\nu}$
be deduced by an action principle, which constrains
the form of the equation sufficiently to determine $\chi^{\mu\nu}$ and
the equation for $h_{\mu\nu}$ to $O(h^2)$.

But now the new $T^{\mu\nu} + \chi^{\mu\nu}$ is only conserved to $O(h \cdot T)$,
so we wash, rinse, and repeat, extending the story to
higher order in h.

This procedure works, but is unwieldy.

At this point, we had better enter Einstein's world,
a world in which spacetime is the main character.