Plane wave solutions

Assume $T_{\mu\nu}=0$.

$$(*)\begin{cases} \partial_\alpha \partial^\alpha h_{\mu\nu}=0 & \text{linearized Einstein equations in harmonic gauge} \\ \partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h & \text{harmonic gauge conditions.} \end{cases}$$

We look for plane wave solutions to the set of equations (*).

Plane wave solutions have the form

$$\boxed{h_{\mu\nu}(x) = \epsilon_{\mu\nu}\exp(ik\cdot x) + \epsilon_{\mu\nu}^*\exp(-ik\cdot x)}$$

where $k\cdot x \equiv k_\mu x^\mu = k^\mu x_\mu$, $\epsilon_{\mu\nu}$ = polarization tensor, can be complex.

$h_{\mu\nu}=h_{\nu\mu} \Leftrightarrow \boxed{\epsilon_{\mu\nu}=\epsilon_{\nu\mu}}$

Consider derivatives of $\exp(ik\cdot x)$:

$$\begin{aligned} \partial_\alpha \exp(ik_\mu x^\mu) &= \frac{\partial}{\partial x^\alpha}\exp(ik_\mu x^\mu) \\ &= ik_\mu \frac{\partial x^\mu}{\partial x^\alpha}\exp(ik_\nu x^\nu) \\ &= ik_\mu \delta^\mu{}_\alpha \exp(ik\cdot x) \\ &= ik_\alpha \exp(ik\cdot x) \end{aligned}$$

Similarly, $\partial_\alpha\partial^\alpha \exp(ik\cdot x) = -k_\alpha k^\alpha \exp(ik\cdot x)$.

Then the equations (*) imply

$$\begin{cases} -k_\alpha k^\alpha h_{\mu\nu} = 0 \Rightarrow \boxed{-k_\alpha k^\alpha = 0} \text{ if } h_{\mu\nu}\neq 0 \\ k_\mu \epsilon^\mu{}_\nu = \frac{1}{2}k_\nu \epsilon^\mu{}_\mu \end{cases}$$

Consider the gauge transformation $h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

with $\xi^\mu(x) = -i\varepsilon^\mu \exp(ik\cdot x) + i\varepsilon^{\mu *}\exp(-ik\cdot x)$.

This is equivalent to $\epsilon_{\mu\nu}\to \epsilon_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$.

Under this class of gauge transformations,

$$K_\mu \epsilon^\mu{}_\nu \rightarrow K_\mu \epsilon^\mu{}_\nu + \underbrace{\cancel{K_\mu K^\mu}}_{0} \varepsilon_\nu + \underbrace{(K_\mu \varepsilon^\mu)}_{K\cdot\varepsilon} K_\nu$$

$$\tfrac{1}{2} K_\nu \epsilon^\mu{}_\mu \rightarrow \tfrac{1}{2} K_\nu \epsilon^\mu{}_\mu + \tfrac{1}{2}\cdot 2 K_\nu (K^\mu \varepsilon_\mu)$$
$$= \tfrac{1}{2} K_\nu \epsilon + K_\nu (K\cdot\varepsilon)$$

$$\underbrace{0 = K_\mu \epsilon^\mu{}_\nu - \tfrac{1}{2} K_\nu \epsilon^\mu{}_\mu}_{\text{harmonic gauge condition}} \rightarrow K_\mu \epsilon^\mu{}_\nu - \tfrac{1}{2} K_\nu \epsilon^\mu{}_\mu$$

Number of independent solutions:
For each K^μ satisfying $K_\mu K^\mu = 0$,

$\epsilon_{\mu\nu}$ —	symmetric 4x4 matrix	10 components
	harmonic gauge condition	-4
	remaining gauge freedom	-4
		$\boxed{\text{2 independent polarizations}}$

— like E&M!

Example: Wave traveling in x^3-direction.

$$K^1 = K^2 = 0, \quad K^3 = K^0 \equiv K > 0.$$

Harmonic condition:
$$\begin{cases} K^3\epsilon_{31} + K^0\epsilon_{01} = K^3\epsilon_{32} + K^0\epsilon_{02} = 0 \\ K^3\epsilon_{33} + K^0\epsilon_{03} = -(K^3\epsilon_{30} + K^0\epsilon_{00}) = \tfrac{1}{2}K^3(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}-\epsilon_{00}) \end{cases}$$

$$K^3 = K^0 = K \Rightarrow \begin{cases} \epsilon_{31} + \epsilon_{01} = \epsilon_{32} + \epsilon_{02} = 0 \\ \epsilon_{33} + \epsilon_{03} = -(\epsilon_{30} + \epsilon_{00}) = \tfrac{1}{2}(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}-\epsilon_{00}) \end{cases}$$

$$\Rightarrow \boxed{\begin{array}{ll} \epsilon_{01} = -\epsilon_{31}, & \epsilon_{02} = -\epsilon_{32} \\ \epsilon_{22} = -\epsilon_{11}, & \epsilon_{03} = -\tfrac{1}{2}(\epsilon_{33}+\epsilon_{00}) \end{array}}$$

$\epsilon_{01}, \epsilon_{02}, \epsilon_{22}, \epsilon_{03}$ dependent on other polarizations.

Residual gauge freedom:
$$\left.\begin{aligned} \epsilon_{13} &\to \epsilon_{13} + k\,\varepsilon_1 \\ \epsilon_{23} &\to \epsilon_{23} + k\,\varepsilon_2 \\ \epsilon_{33} &\to \epsilon_{33} + 2k\,\varepsilon_3 \\ \epsilon_{00} &\to \epsilon_{00} - 2k\,\varepsilon_0 \end{aligned}\right\}$$
Can choose $\boxed{\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = \epsilon_{00} = 0}$
$\to$ unphysical polarizations.

$\Rightarrow$ Only two components ($\epsilon_{11}, \epsilon_{12}$) have independent physical significance.

$$\left.\begin{aligned} \epsilon_{01} &= -\epsilon_{31} = -\epsilon_{13} = 0 \\ \epsilon_{02} &= -\epsilon_{32} = -\epsilon_{23} = 0 \\ \epsilon_{03} &= -\tfrac{1}{2}(\epsilon_{33} + \epsilon_{00}) = 0 \end{aligned}\right\}$$
from harmonic condition and above gauge choice.

The polarization tensor in this gauge takes the form

$$\boxed{\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}}$$
for some ϵ_+, $\epsilon_\times$

Notice that in this gauge, $\boxed{k^\mu \epsilon_{\mu\nu} = 0 \leftarrow \text{transverse}}$
and $\boxed{\epsilon^\mu{}_\mu = 0 \leftarrow \text{traceless}}$

This is called <u>transverse, traceless gauge.</u>

<u>Caution:</u> Notice that we used the equations of motion with $T_{\mu\nu} = 0$ in order to deduce $k_\mu k^\mu = 0$, which led to $k^3 = k^0$ here. If $T_{\mu\nu} \neq 0$, we might not be able to simultaneously satisfy the equations of motion (i.e. the linearized Einstein eqs.) and the transverse + traceless conditions on $h_{\mu\nu}$.

Helicity of Gravitational Waves

Consider a rotation by angle θ about the x^3-axis,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}, \quad (\Lambda^{-1})^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$$

The polarization tensor transforms as

$$\epsilon_{\mu\nu} \to \epsilon'_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu \epsilon_{\alpha\beta}$$

Defining $\begin{cases} \epsilon_\pm \equiv \epsilon_{11} \pm i\epsilon_{12} = -\epsilon_{22} \pm i\epsilon_{12} \\ f_\pm \equiv \epsilon_{31} \pm i\epsilon_{32} = -\epsilon_{01} \mp i\epsilon_{02} \end{cases}$,

it is straightforward to check that under the rotation,

$$\epsilon'_\pm = \exp(\pm 2 i\theta)\,\epsilon_\pm \quad \leftarrow \text{helicity } \pm 2$$

$$f'_\pm = \exp(\pm i\theta)\, f_\pm \quad \leftarrow \text{helicity } \pm 1$$

$$\epsilon'_{33} = \epsilon_{33}, \; \epsilon'_{00} = \epsilon_{00} \quad \leftarrow \text{helicity } 0$$

Any plane wave which transforms as $\psi' = e^{ih\theta}\psi$ under a rotation by θ about the direction of motion is said to have <u>helicity h</u>.

In our analysis of plane wave solutions for $h_{\mu\nu}$, we chose $k^1 = k^2 = 0$, so motion is in the x^3 direction.

We also found that the physical components of $\epsilon_{\mu\nu}$ were ϵ_{11} and ϵ_{12}, which could be replaced by the linear combinations $\epsilon_\pm$.

Hence, <u>gravitational waves</u> are decomposed into helicity $\pm 2, \pm 1, 0$ parts, but only the <u>helicity ± 2</u> parts are physical.

As usual, comparison with E&M is useful.

Source-free Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

In Lorenz gauge $\boxed{\partial_\alpha A^\alpha = 0}$, $\partial_\mu F^{\mu\nu} = \boxed{\partial_\mu \partial^\mu A^\nu = 0}$

Gauge transformations $A^\mu \to A^\mu + \partial^\mu \theta$ with $\partial_\mu \partial^\mu \theta = 0$ preserve the Lorenz gauge condition.

$$\#\text{ propagating degrees of freedom} = 4 - \underset{\substack{\text{Lorenz gauge}\\ \text{condition}}}{1} - \underset{\substack{\text{residual gauge}\\ \text{freedom}}}{1} = 2$$

Under a rotation, $A_\mu \to (\Lambda^{-1})^\alpha{}_\mu A_\alpha$

One factor of Λ^{-1} (rather than two in transformation of $h_{\mu\nu}$) $\longrightarrow$ physical plane waves have helicity ± 1 in electromagnetism.

(Exercise)

Motion of particles

To consider motion of particles in a background of $h_{\mu\nu}$, we need to identify how $h_{\mu\nu}$ appears in the metric tensor $g_{\mu\nu}$. A natural guess is $\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$, as in the Newtonian limit studied earlier.

Particles then follow the geodesic equation with this $g_{\mu\nu}$,

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 .$$

Motion of particles in a gravitational wave

Zee IX.4

Consider a particle initially at rest, $\frac{dx^0}{d\tau} = 1$, $\frac{d\vec{x}}{d\tau} = 0$.
The particle follows the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00} \approx 0 \quad \text{near the initial instant}$$

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_0 h_{0\lambda} + \partial_0 h_{0\lambda} - \partial_\lambda h_{00}\right) + \mathcal{O}(h^2)$$

For a plane wave in transverse-traceless gauge, $\epsilon_{0\lambda} = \epsilon_{\lambda 0} = 0$.

$$\rightarrow \Gamma^\mu_{00} \approx 0.$$

$$\frac{d^2x^\mu}{d\tau^2} \approx 0. \leftarrow$$ particle at rest remains at rest, at least for short times.

If a particle does not respond to a passing gravitational wave, then how would such a wave be detected?
Answer: Consider a collection of particles.

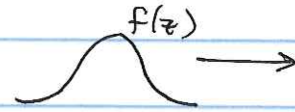
A physical gravitational wave would be in the form of a wavepacket, ie. a superposition of plane waves.
For example, consider a superposition of plane waves in the x^3-direction

$$k^\mu \sim (k, 0, 0, k)$$

$$h_{\mu\nu} = \int dk\ \tilde{f}(k)\, e^{ik(z-t)}\epsilon_{\mu\nu}(k) + \text{c.c.} \quad \text{complex conjugate}$$

Suppose $\epsilon_{\mu\nu}(k) = \epsilon_{\mu\nu}$ independent of k.

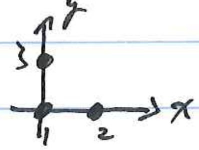
Then $h_{\mu\nu}$ has the form $h_{\mu\nu} \equiv f(z-t)\,\epsilon_{\mu\nu} + \text{c.c.}$



In transverse-traceless gauge,

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Suppose there are three particles in the x-y plane:



$\vec{x}_1 = (0, 0, 0)$
$\vec{x}_2 = (a, 0, 0)$
$\vec{x}_3 = (0, a, 0)$

Assume a is small compared to the width of the wavepacket.

The proper distance between these points is:

$$(\Delta s_{12})^2 = g_{xx}\, a^2 = (1 + h_{xx})\, a^2$$

$$\Delta s_{12} = a\sqrt{1 + h_{xx}} \approx a\left(1 + \tfrac{1}{2}\left(f(z-t)\epsilon_+ + c.c.\right)\right)$$

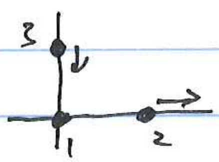
$$\boxed{= a\left(1 + \mathrm{Re}\left(f(z-t)\epsilon_+\right)\right)}$$

$$(\Delta s_{13})^2 = g_{yy}\, a^2 = (1 + h_{yy})\, a^2$$

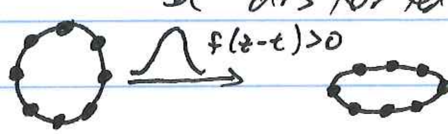
$$\Delta s_{13} \approx a\left(1 - \tfrac{1}{2}\left(f(z-t)\epsilon_+ + c.c.\right)\right)$$

$$\boxed{= a\left(1 - \mathrm{Re}\left(f(z-t)\epsilon_+\right)\right)}$$

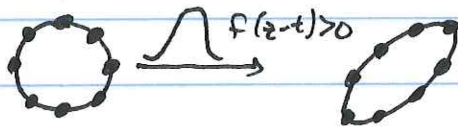
As the distance between 1 and 3 shrinks, the distance between 1 and 2 grows, and vice versa.



A circular distribution of particles would be distorted into an ellipsoidal shape:



$\epsilon_+ < 0,\ \epsilon_\times = 0$



$\epsilon_+ = 0,\ \epsilon_\times < 0$

This distortion is the basis of gravitational wave searches.