cf. Zee 3.6, Feynman Ch. 3,4

Dynamics of Gravity

We have seen that a consequence of the Einstein Equivalence Theorem is that the effects of gravitation on a freely-falling particle are encoded in the metric tensor $g_{\mu\nu}$.
We have also seen that in the Newtonian limit, $g_{\mu\nu}$ is related to the gravitational potential ϕ via $g_{00} = -(1+2\phi)$.

We now turn to the question of what determines the dynamics of gravity, i.e. how to determine $g_{\mu\nu}(\vec{x},t)$ more generally.

Scalar Gravity?

One possibility is that even away from the Newtonian limit, gravity is entirely encoded in a potential $\phi(\vec{x},t)$, for example with $g_{\mu\nu} = \eta_{\mu\nu}(1+2\phi)$.

Problems: 1) If ϕ satisfies the Newtonian relation $\nabla^2\phi = 4\pi G\rho$, then changes in ρ affect ϕ instantaneously at a distance, in seeming conflict with special relativity.

This problem is easy to fix. Suppose that instead, the scalar potential (or scalar field) $\phi(\vec{x},t)$ satisfies

$$\boxed{-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = 4\pi G\rho}$$

When $\rho=0$ this is the wave equation for ϕ, which is consistent with special relativity. $(\partial_\mu\partial^\mu\phi = 0)$

However, 2) $\phi(\vec{x},t)$ is a scalar field, while the mass density $\rho(\vec{x},t)$ transforms nontrivially under Lorentz transformations.

The density $\rho(\vec{x},t)$ is a component of a tensor in special relativity, the stress-energy tensor, which describes the density and flux of energy and momentum.

The stress-energy tensor in special relativity is the conserved current associated with spacetime translations via Noether's theorem. We denote the stress-energy tensor by $T_{\mu\nu}$. Conservation implies $\boxed{\partial_\mu T^{\mu\nu} = 0,}$

i.e.

$$\frac{\partial}{\partial t} T^{0\nu} + \sum_{i=1}^{3} \frac{\partial}{\partial x^i} T^{i\nu} = 0$$

As long as $T^{\mu\nu}$ falls off sufficiently quickly at ∞, the conservation law implies four time-independent quantities, one for each value of the index ν.

Consider

$$\int_V d^3x \left[\frac{\partial}{\partial t} T^{0\nu} + \sum_{i=1}^{3} \frac{\partial}{\partial x^i} T^{i\nu} \right] = 0$$

, integrated over some region V

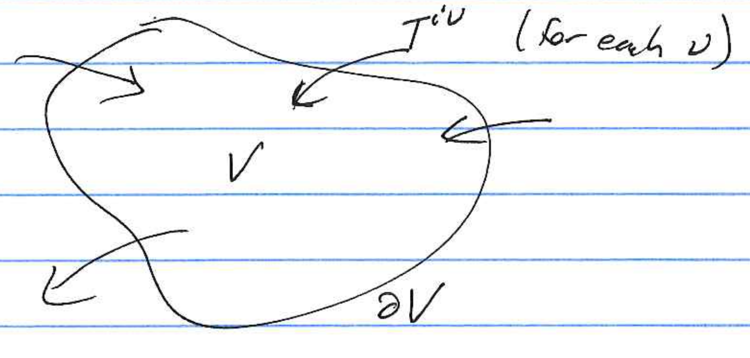
$$\frac{d}{dt} \int_V d^3x \, T^{0\nu} = - \int_V d^3x \sum_{i=1}^{3} \frac{\partial}{\partial x^i} T^{i\nu}$$

$$= - \int_{\partial V} d^2x \, n_i T^{i\nu}$$

, where ∂V is the boundary of V and n_i = unit vector normal to boundary of V.

(compare with $\int_V d^3x \, \nabla \cdot \vec{J} = \int_{\partial V} d^2x \, \hat{n} \cdot \vec{J}$ — Gauss' law)

The surface integral $\int_{\partial V} d^2x\, n_i T^{i\nu}$ represents the flux of $T^{i\nu}$ through the surface ∂V.



Taking the region V to fill all space, as long as $T^{i\nu}$ falls off quickly enough at ∂V,

$$\frac{d}{dt}\left(\int d^3x\, T^{0\nu}\right) = 0.$$

Defining $P^\nu \equiv \int d^3x\, T^{0\nu}$,

P^0 is the conserved quantity associated with time-translation invariance, i.e. the energy E. Hence, T^{00} is the energy density, $\boxed{T^{00} = \rho}$.

P^i, $i=1,2,3$ are the conserved quantities associated with spatial translation invariance in each of the three orthogonal directions, i.e. the spatial momentum $\vec{p}$. Hence, T^{0i} is the momentum density.

The components T^{ij} give the flux of the i^{th} component of momentum across a surface $x^j = $ const. For a fluid, T^{12}, T^{13}, T^{23} are the components of the shear stress, and T^{11}, T^{22}, T^{33} the pressure.

In order to salvage the scalar theory of gravity, we could guess that the correct equation for $\phi(\vec{x},t)$ is

$$\boxed{\partial_\mu \partial^\mu \phi = -T^\mu{}_\mu \cdot 4\pi G}$$ $-$ve sign because $T^0{}_0 = -T^{00} = -\rho$.

This has the features that both sides of the equation are Lorentz scalars if ϕ is a scalar field, and as long as $|T^{00}| \gg |T^{ii}|$ for all $i=1,2,3$, as is typically the case in the Newtonian limit, then the above equation for ϕ has the correct Newtonian limit.

The problem with this theory of gravity is 3) it doesn't agree with observation. This theory predicts incorrect bending of light by the sun, precession of the perihelion of Mercury, ...

At this stage we give up on our attempt to formulate a scalar theory of gravity.

Vector Gravity?

How about a vector field describing gravity, like in E&M?

Problem: 1) Like charges repel if the interaction is mediated by a vector field. In gravity it seems that things attract one another.

So, we give up on a vector theory of gravity.

Tensor Gravity

The next-simplest possibility is that gravity is mediated by a two-index tensor field $h_{\mu\nu}(\vec{x},t)$. We can anticipate that the source for $h_{\mu\nu}$, i.e. the term replacing $\rho(\vec{x},t)$ on the right-hand-side of the equation for $h_{\mu\nu}$, is the stress-energy tensor $T^{\mu\nu}$.

The symmetric and antisymmetric parts of $h_{\mu\nu}$ (as a matrix) are preserved by Lorentz transformations.
For example, if $h_{\mu\nu} = h_{\nu\mu}$, then

$$h'_{\alpha\beta} = (\Lambda^{-1})^{\mu}{}_{\alpha} (\Lambda^{-1})^{\nu}{}_{\beta}\, h_{\mu\nu} = (\Lambda^{-1})^{\nu}{}_{\beta} (\Lambda^{-1})^{\mu}{}_{\alpha}\, h_{\nu\mu} = h'_{\beta\alpha}.$$

Similarly, if $h_{\mu\nu} = -h_{\nu\mu}$, then

$$h'_{\alpha\beta} = (\Lambda^{-1})^{\mu}{}_{\alpha} (\Lambda^{-1})^{\nu}{}_{\beta}\, h_{\mu\nu} = (\Lambda^{-1})^{\nu}{}_{\beta} (\Lambda^{-1})^{\mu}{}_{\alpha} (-h_{\nu\mu}) = -h'_{\beta\alpha}$$

The stress-energy tensor can be chosen to be symmetric $T_{\mu\nu} = T_{\nu\mu}$.
Hence, we will guess that we only need the symmetric part of $h_{\mu\nu}$ so we assume that $h_{\mu\nu} = h_{\nu\mu}$.

To determine a reasonable equation for $h_{\mu\nu}$ we are guided by a few principles:

1) Lorentz covariance
2) Symmetry $T_{\mu\nu} = T_{\nu\mu}$
3) conservation $\partial_\mu T^{\mu\nu} = 0$
4) Simplicity: Assume the equation is linear in $h_{\mu\nu}$ and its derivatives.
5) Agrees with experiment and observation.

If the right-hand side of the equation is proportional to $T_{\mu\nu}$, the left-hand side should be composed of terms each of which is a symmetric rank-2 tensor.

Terms with the minimal number of derivatives are:

No derivatives: $h_{\mu\nu}$, $\eta_{\mu\nu} h_\alpha{}^\alpha \equiv \eta_{\mu\nu} h$

One derivative: none.

Two derivatives: $\partial_\alpha \partial^\alpha h_{\mu\nu}$

$\partial_\mu \partial_\nu h$

$\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}$

$\eta_{\mu\nu} \partial_\alpha \partial^\alpha h$

$\eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta}$

Suppose the equation for $h_{\mu\nu}$ has the form

$$\boxed{\begin{aligned} &a\, \partial_\alpha \partial^\alpha h_{\mu\nu} + b\, \partial_\mu \partial_\nu h + c\left(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}\right) \\ &\quad + d\, \eta_{\mu\nu} \partial_\alpha \partial^\alpha h + e\, \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} = -\lambda T_{\mu\nu} \end{aligned}}$$

for some constants a, b, c, d, e, λ.

We don't add terms without derivatives because we have seen in our discussion of Newtonian gravity that if we modify the equation $\nabla^2 \phi = 4\pi G\rho$ to $\nabla^2\phi - m^2\phi = 4\pi G\rho$, then solutions fall exponentially away from a localized source ρ. Gravity is a long-ranged force, so we assume that "mass terms" linear in $h_{\mu\nu}$ vanish.

For particular relations between a, b, c, d, e, the conservation law $\partial^\mu T_{\mu\nu} = 0$ will follow as a consequence of the equation for $h_{\mu\nu}$.

Acting on the equation with ∂^μ, we get

$$a\,\partial_\alpha\partial^\alpha\partial^\mu h_{\mu\nu} + b\,\partial^\mu\partial_\mu\partial_\nu h + c\left(\partial^\mu\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial^\mu\partial_\nu\partial^\alpha h_{\alpha\mu}\right)$$
$$+ d\,\partial_\nu\partial_\alpha\partial^\alpha h + e\,\partial_\nu\partial_\alpha\partial_\beta h^{\alpha\beta} = -\lambda\,\partial^\mu T_{\mu\nu} = 0$$

$$\partial_\alpha\partial^\alpha\partial^\mu h_{\mu\nu}\,(a+c) + \partial^\mu\partial_\mu\partial_\nu h\,(b+d) + \partial^\mu\partial_\nu\partial^\alpha h_{\alpha\mu}\,(c+e) = 0$$

This is automatically satisfied if $a = e = -c$ and $b = -d$.

Choosing $a = 1$, the equation for $h_{\mu\nu}$ becomes

$$\boxed{\begin{aligned}\partial_\alpha\partial^\alpha h_{\mu\nu} - \left(\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu}\right) + \eta_{\mu\nu}\,\partial_\alpha\partial_\beta h^{\alpha\beta}& \\ + b\left(\partial_\mu\partial_\nu h - \eta_{\mu\nu}\,\partial_\alpha\partial^\alpha h\right) &= -\lambda T_{\mu\nu}\end{aligned}}$$

<u>Gauge Invariance</u>

Assume $h_{\mu\nu}$ has a part of the form $h_{\mu\nu} \Rightarrow \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$ for some vector field $\varepsilon_\mu(x)$.

The left-hand side of the equation for $h_{\mu\nu}$ then includes

$$\partial_\alpha\partial^\alpha\left(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right) - \partial_\mu\partial^\alpha\left(\partial_\alpha\varepsilon_\nu + \partial_\nu\varepsilon_\alpha\right) - \partial_\nu\partial^\alpha\left(\partial_\alpha\varepsilon_\mu + \partial_\mu\varepsilon_\alpha\right)$$
$$+ \eta_{\mu\nu}\,\partial_\alpha\partial_\beta\left(\partial^\alpha\varepsilon^\beta + \partial^\beta\varepsilon^\alpha\right) + b\,\partial_\mu\partial_\nu\left(\partial_\alpha\varepsilon^\alpha + \partial_\alpha\varepsilon^\alpha\right)$$
$$- b\,\eta_{\mu\nu}\,\partial_\alpha\partial^\alpha\left(\partial_\beta\varepsilon^\beta + \partial_\beta\varepsilon^\beta\right)$$
$$= \partial_\alpha\partial^\alpha\cancel{\partial_\mu\varepsilon_\nu(1-1)} + \partial_\alpha\partial^\alpha\cancel{\partial_\nu\varepsilon_\mu(1-1)} + \partial_\mu\partial_\nu\partial_\alpha\varepsilon^\alpha(-1-1+2b)$$
$$+ \eta_{\mu\nu}\,\partial_\alpha\partial^\alpha\partial_\beta\varepsilon^\beta(1+1-2b)$$
$$= 0 \text{ if } b = 1.$$

With the choice $b=1$, the equation for $h_{\mu\nu}$ is invariant under $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$.

For any solution to the equation for $h_{\mu\nu}$, there are then a continuum of other solutions, one for every $\varepsilon_\mu(x)$.

This is a redundancy in the description of the field $h_{\mu\nu}$, a gauge invariance much like in the relativistic description of electromagnetism.

In E&M, the equation for the vector field A_μ is

$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 4\pi J_\nu$, where J_ν is the 4-vector current.

The term in parentheses is the field strength tensor

$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. It is antisymmetric, and its nonvanishing components are the components of $\vec{E}$ and $\vec{B}$. (Note that $\partial^\nu J_\nu = 0$ follows from the eqn. for A_μ.)

The equation of motion for A_μ is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta(x)$ for any function $\theta(x)$.

This is a redundancy in the description of the electromagnetic field in terms of A_μ, and a gauge condition is required in order to uniquely specify a solution for A_μ.

The redundancy reduces the number of propagating degrees of freedom in A_μ. There are 4 components in A_μ, but only 2 propagating degrees of freedom, the two helicities of the photon

If we can describe gravitation with fewer degrees of freedom, then we are compelled to at least try.

We are thereby led to the <u>linearized Einstein equations</u>:

$$\boxed{\partial_\alpha \partial^\alpha h_{\mu\nu} - (\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu}) + \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\alpha \partial^\alpha h = -\lambda T_{\mu\nu}}$$