Just as in electrodynamics, we can choose to fix a gauge condition on $h_{\mu\nu}$ in order to make equations look simpler.

For example, we may insist that $h_{\mu\nu}$ satisfy,

$$\boxed{\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h} \quad \text{Harmonic gauge.}$$

To see that we can make this choice, assume

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h = f^\nu(x) \neq 0.$$

Let $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ (gauge transformation)

Then $\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h \to \underbrace{\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h}_{f_\nu(x)} + \partial_\mu\partial^\mu \xi_\nu + \cancel{\partial_\mu\partial_\nu \xi^\mu} - \cancel{\frac{1}{2}\cdot 2\partial_\nu\partial_\mu \xi^\mu}$

$$= 0 \text{ if } \partial_\mu\partial^\mu \xi_\nu = -f_\nu(x),$$

which can be solved for $\xi_\nu(x)$.

Note that the harmonic gauge condition does not completely fix the gauge, because we can make a further gauge transformation with $\partial_\mu\partial^\mu \xi^\nu = 0$ (the wave equation) while maintaining the harmonic gauge condition.

In the harmonic gauge, the linearized Einstein equations become:

$$\partial_\alpha\partial^\alpha h_{\mu\nu} - \frac{1}{2}\cancel{\partial_\mu\partial_\nu h} - \frac{1}{2}\cancel{\partial_\nu\partial_\mu h} + \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial^\alpha h + \cancel{\partial_\mu\partial_\nu h} - \eta_{\mu\nu}\partial_\alpha\partial^\alpha h = -\lambda T_{\mu\nu}$$

or,

$$\boxed{\partial_\alpha\partial^\alpha h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial^\alpha h = -\lambda T_{\mu\nu}}$$

linearized Einstein equations in harmonic gauge.

We can simplify the appearance of the linearized Einstein equations still further.

Given a 2-index tensor $A_{\mu\nu}$, define

$$\boxed{\bar{A}_{\mu\nu} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{2}\eta_{\mu\nu} A^{\sigma}{}_{\sigma}}$$

For a symmetric tensor like $h_{\mu\nu}$,

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h}$$

Also note, $\bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$. (Exercise)

The linearized Einstein eqs. in harmonic gauge may be written

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\lambda T_{\mu\nu}}$$

Taking the trace: $\partial_\alpha \partial^\alpha \left(h^\mu{}_\mu - \frac{1}{2}\delta^\mu{}_\mu h\right) = -\lambda T^\mu{}_\mu$

or, $-\partial_\alpha \partial^\alpha h = -\lambda T$

Adding $\frac{1}{2}\eta_{\mu\nu}\partial_\alpha \partial^\alpha h$ to the linearized Einstein eqs gives

$$\partial_\alpha \partial^\alpha h_{\mu\nu} = -\lambda T_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\partial_\alpha \partial^\alpha h = -\lambda T_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\lambda T$$

i.e.,

$$\boxed{\partial_\alpha \partial^\alpha h_{\mu\nu} = -\lambda \bar{T}_{\mu\nu}}$$