

Comparison between Stationarization of Proper Time and Lagrangian Formalism

We have seen that the proper time elapsed along a trajectory is stationarized for freely falling trajectories (in the absence of external forces).

In classical mechanics, the action functional is stationarized for trajectories that satisfy the equations of motion. There is a relation between these two stationarization principles,

In the weak-field Newtonian limit the metric is related to the gravitational potential $\phi(\vec{x})$, as we have seen, via $g_{00} \approx -(1+2\phi(\vec{x}))$.

We consider ϕ as small (weak field), and assume $|\frac{d\vec{x}}{dt}| \ll 1$ (slow compared to light).

Consider trajectories which begin at \vec{x}_A at time t_A , and end at \vec{x}_B at time t_B . The proper time is

$$\begin{aligned} T(A \rightarrow B) &\approx \int_A^B \sqrt{dt^2(1+2\phi) - d\vec{x}^2} + \text{higher order in } \phi, \frac{d\vec{x}}{dt}. \\ &= \int_{t_A}^{t_B} dt \sqrt{(1+2\phi) - \left(\frac{d\vec{x}}{dt}\right)^2} \\ &\approx \int_{t_A}^{t_B} dt \left(1 + \phi - \frac{1}{2} \left(\frac{d\vec{x}}{dt}\right)^2\right) \quad (\text{expanding the square root about 1.}) \end{aligned}$$

Multiplying by $-m$, where m is the particle's mass,

$$-m T(A \rightarrow B) \approx \int_{t_A}^{t_B} dt \left(\frac{1}{2} m \left(\frac{d\vec{x}}{dt} \right)^2 - m\phi - m \right)$$

If $T(A \rightarrow B)$ is stationarized, so is $-m T(A \rightarrow B)$.

Up to the addition of the constant $-m(t_B - t_A)$, we recognize the action for a particle moving in a gravitational potential $\phi(\vec{x})$:

$$S \equiv -m T(A \rightarrow B) \approx \int_{t_A}^{t_B} L dt + \text{constant}$$

$\uparrow -m(t_B - t_A)$

$$\text{where } L = \frac{1}{2} m \left(\frac{d\vec{x}}{dt} \right)^2 - m\phi(\vec{x})$$

For example, for a particle in a uniform gravitational field with gravitational acceleration $\vec{g} = g\hat{z}$, $\phi = gz$, and

$$L = \frac{1}{2} m \left(\frac{d\vec{x}}{dt} \right)^2 - mgz,$$

$$T(A \rightarrow B) \approx \int_{t_A}^{t_B} \left[(1 + 2gz) - \left(\frac{d\vec{x}}{dt} \right)^2 \right]^{1/2}$$

The proper time is larger if the trajectory spends time at larger z , but in order to reach larger z , $\left(\frac{dz}{dt} \right)^2$ must also be larger somewhere along the trajectory, which reduces T .

The parabolic trajectory which maximizes T is a compromise between minimizing $\left(\frac{d\vec{x}}{dt} \right)^2$ while maximizing z .