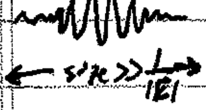
Power Radiated in Gravitational Waves

Consider a plane wave $h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik\cdot x} + \epsilon^*_{\mu\nu} e^{-ik\cdot x}$.
The energy-momentum "tensor" in the wave pulse is given by $t_{\mu\nu} = \frac{1}{8\pi G_N}\left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \left(R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)}\right)\right]$

The expansion of $t_{\mu\nu}$ in $\epsilon_{\mu\nu}$ is messy, but simplifies if we consider the spatial average over distances $\gg \frac{1}{|k|}$, $\langle t_{\mu\nu}\rangle$. Then we can use $\langle e^{\pm 2ik\cdot x}\rangle = 0$, which eliminates a number of terms.



$\langle t_{\mu\nu}\rangle$ = spatial average over pulse.

Using $R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^\lambda{}_\lambda - \partial_\lambda\partial_\nu h^\lambda{}_\mu - \partial_\lambda\partial_\mu h^\lambda{}_\nu + \partial_\lambda\partial^\lambda h_{\mu\nu}\right)$,

$$R^{(1)} = \partial_\lambda\partial^\lambda h - \partial_\lambda\partial_\rho h^{\lambda\rho}$$

$$\boxed{\begin{aligned} R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = \frac{1}{2}\Big(&\partial_\mu\partial_\nu h - \partial_\lambda\partial_\nu h^\lambda{}_\mu - \partial_\lambda\partial_\mu h^\lambda{}_\nu + \partial_\lambda\partial^\lambda h_{\mu\nu} \\ &- \eta_{\mu\nu}\partial_\lambda\partial^\lambda h^\rho{}_\rho + \eta_{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho}\Big)\end{aligned}}$$

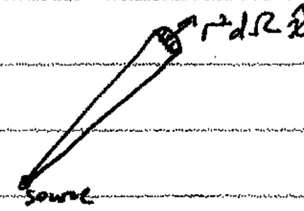
Comparing with our earlier analysis of plane waves, $R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = 0$, which confirms the linearized approximation of the vacuum Einstein eqns.

$$\epsilon_{\mu\nu}^* \epsilon^{\mu\nu} = \frac{16G^2}{r^2}\Big(\tilde{T}_{\mu\nu}^* \tilde{T}^{\mu\nu} - \frac{1}{2}\tilde{T}^* \eta_{\mu\nu}\tilde{T}^{\mu\nu} - \frac{1}{2}\tilde{T}_{\mu\nu}^*\eta^{\mu\nu}\tilde{T}$$
$$+ \frac{1}{4}\underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{4}\tilde{T}^*\tilde{T}\Big)$$

$$= \frac{16G^2}{r^2}\tilde{T}^*_{\mu\nu}\tilde{T}^{\mu\nu}$$

$$|\epsilon^\lambda{}_\lambda|^2 = \frac{16G^2}{r^2}\Big|\tilde{T}^\lambda{}_\lambda - \frac{1}{2}\underbrace{\delta^\lambda{}_\lambda}_{4}\tilde{T}\Big|^2 = \frac{16G^2}{r^2}|\tilde{T}|^2$$

$$\rightarrow \boxed{\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G_N}\cdot\frac{16G_N^2}{r^2}\left(\tilde{T}^*_{\mu\nu}\tilde{T}^{\mu\nu} - \frac{1}{2}|\tilde{T}|^2\right)}$$



Away from sources, $\partial_\mu t^{\mu\nu} = 0$

$$\partial_0 t^{00} = -\partial_i t^{i0}$$

$\int d^3x$: $\quad E = \int d^3x\, t^{00}$

Gauss' law

$$P \equiv -\frac{dE}{dt} = +\int_V d^3x\,\partial_i t^{i0} = +\int_{\partial V} d\Omega\, r^2\hat{x}^i t_i{}^0$$

∂V ← boundary of V

$P =$ radiated power in gravitational radiation from vol. V

$$\frac{dP}{d\Omega} = r^2\hat{x}^i\langle t^{i0}\rangle$$
$$= \frac{r^2(\vec{k}\cdot\hat{x})k^0}{16\pi G_N}\cdot\frac{16G_N^2}{r^2}\left(\tilde{T}^*_{\mu\nu}\tilde{T}^{\mu\nu} - \frac{1}{2}|\tilde{T}|^2\right)$$

$$= \frac{G_N\omega^2}{\pi}\left(\tilde{T}^*_{\mu\nu}(\vec{k},\omega)\,\tilde{T}^{\mu\nu}(\vec{k},\omega) - \frac{1}{2}|\tilde{T}^\lambda{}_\lambda(\vec{k},\omega)|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi}\Big(\frac{1}{2}\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l D^{*ij}D^{kl} - 2\hat{x}_j\hat{x}_k D^{*ji}D^{ki}$$

$$+ D^{*ij}D^{ij} + \frac{1}{2}\hat{x}_k\hat{x}_l D^{*kl}D^{ii} + \frac{1}{2}\hat{x}_k\hat{x}_l D^{*ii}D^{kl}$$

$$- \frac{1}{2}D^{*ii}D^{jj}\Big)$$

Total Power radiated: $P = \int d\Omega \frac{dP}{d\Omega} = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\, \frac{dP}{d\Omega}$

Use $\int d\varphi\, d(\cos\theta) = 4\pi$

$\int d\varphi\, d(\cos\theta)\, \hat{x}_i\hat{x}_j = \frac{4}{3}\pi\,\delta_{ij}$

$\int d\varphi\, d(\cos\theta)\, \hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right)$

$$\Rightarrow \boxed{P = \frac{2G\omega^6}{5}\left(D^*_{ij}(\omega)D_{ij}(\omega) - \frac{1}{3}|D_{ii}(\omega)|^2\right)}$$

This is the total power radiated at one frequency in the quadrupole approximation $\omega R \ll 1$

$$D^*_{ij} D_{ij} = \left(\frac{MR^2}{2}\right)^2 (1+1+1+1) = M^2 R^4$$

$$D^i_i = D_{xx} + D_{yy} = 0$$

Power $\mathcal{P} = \frac{2G}{5}(2\Omega)^6 M^2 R^4 = \frac{128}{5} G M^2 R^4 \Omega^6$

$$\Omega^2 = \frac{GM}{4R^3} \Rightarrow \boxed{\mathcal{P} = \frac{2}{5}\frac{G^4 M^5}{R^5}}$$

Power radiated in gravitational radiation

$E \propto -T^{-2/3}$ for Newtonian orbits

$$\frac{1}{T}\frac{dT}{dt} = -\frac{3}{2}\frac{1}{E}\frac{dE}{dt} = \frac{3}{2}\frac{1}{E}\mathcal{P}$$

1974 Hulse-Taylor Binary Pulsar

-Supernova remnant PSR 1913+16

Change in orbital period measured, Mass and size of orbit known

→ Agrees with predictions of change in period due to energy loss to gravitational radiation!

Note: When comparing with astronomical data, the semimajor axis and other geometric quantities are typically quoted with respect to one of the objects (called the "principal object"). For example, in our example the semimajor axis would have length $2R$.