cf. Weinberg 10.4

Generation of Gravitational Radiation

In the harmonic gauge the equations of motion were

$$\partial_\sigma \partial^\sigma h_{\alpha\beta} = -16\pi G \overline{T}_{\alpha\beta} = -16\pi G\left(T_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} T\right)$$

Weinberg calls this $S_{\alpha\beta}$

Each component of $h_{\alpha\beta}$ satisfies an equation of the form

$\partial_\sigma \partial^\sigma \phi(x) = \rho(x)$, which can be solved using Green functions.

Suppose $G(x-x')$ satisfies $\boxed{\partial_\sigma \partial^\sigma G(x-x') = \delta^4(x-x')}$

Then $\boxed{\phi(x) = \int G(x-x')\,\rho(x')\,d^4x'}$ satisfies the Green

function eqn. To solve the Green function eqn. we Fourier transform,

$$G(x-x') = \int \frac{d^4k}{(2\pi)^4}\,\tilde{G}(k)\,e^{ik\cdot(x-x')}$$

$$\delta^4(x-x') = \int \frac{d^4k}{(2\pi)^4}\,e^{ik\cdot(x-x')}$$

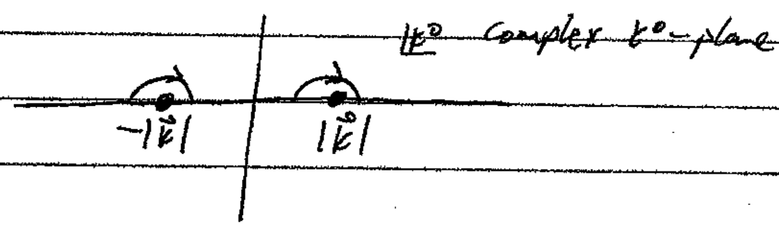
$$\Rightarrow \left(-k^\sigma k_\sigma\right)\tilde{G}(k) = 1$$

$$\boxed{\tilde{G}(k) = -\frac{1}{k^\sigma k_\sigma}} \equiv -\frac{1}{k^2}$$

Then, $$G(x-x') = -\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\,e^{ik\cdot(x-x')}$$

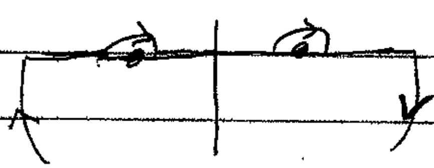
$$G(x-x') = -\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i[k^0(t-t') - \vec{k}\cdot(\vec{x}-\vec{x}')]}}{-(k^0)^2 + \vec{k}^2}$$

The k^0 integration has poles on the real axis, so we need to decide how to integrate around them.



Integrating above the poles gives the retarded Green function, which vanishes if $t<t'$. To see this, assume $t<t'$. The integrand is proportional to $e^{-ik^0(t-t')}$, so with $t<t'$ we can close the contour with a semicircle in the upper half plane. But the integration contour then does not enclose any poles, so it vanishes by Cauchy's theorem.

If $t>t'$ we close the contour in the lower half plane



The residue theorem gives

$$G(x-x') = -\int \frac{d^3k}{(2\pi)^3} \underset{\text{clockwise}}{(-i)} \left\{ \frac{e^{-i[|\vec{k}|(t-t') - \vec{k}\cdot(\vec{x}-\vec{x}')]}}{-2|\vec{k}|} + \frac{e^{-i[-|\vec{k}|(t-t') - \vec{k}\cdot(\vec{x}-\vec{x}')]}}{2|\vec{k}|} \right\} \theta(t-t')$$

where $\theta(t-t') = 1$ if $t>t'$, 0 if $t<t'$.

$$G(x-x') = -\frac{i}{2}\int_0^\infty \frac{dk}{(2\pi)^3}k^2\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\left\{\frac{e^{-ik[t-t'-|\vec{x}-\vec{x}'|\cos\theta]}}{k}\right.$$

$$\left.-\frac{e^{ik[t-t'+|\vec{x}-\vec{x}'|\cos\theta]}}{k}\right\}\theta(t-t')$$

φ integral

$$= \frac{-i(2\pi)}{2(2\pi)^3}\int_0^\infty \frac{dk\,k^2}{k}\left\{\left(\frac{e^{-ik[t-t'-|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}-\frac{e^{-ik[t-t'+|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}\right)\right.$$

$$\left.+\left(-\frac{e^{ik[t-t'+|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}+\frac{e^{ik[t-t'-|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}\right)\right\}$$

$$\times\theta(t-t')$$

Taking $k\to -k$ in the second part of integrals gives

$$G(x-x') = -\frac{1}{8\pi^2}\int_{-\infty}^{\infty} dk\left\{\frac{e^{-ik[t-t'-|\vec{x}-\vec{x}'|]}-e^{-ik[t-t'+|\vec{x}-\vec{x}'|]}}{|\vec{x}-\vec{x}'|}\right\}$$

$$\times\theta(t-t')$$

$$= -\frac{1}{8\pi^2|\vec{x}-\vec{x}'|}\cdot\left\{2\pi\,\delta(t-t'-|\vec{x}-\vec{x}'|)-2\pi\,\delta(t-t'+|\vec{x}-\vec{x}'|)\right\}$$

$$\times\theta(t-t')$$

$$\boxed{G(x-x') = -\frac{1}{4\pi|\vec{x}-\vec{x}'|}\,\delta(t-t'-|\vec{x}-\vec{x}'|)}$$

where we dropped the second term because it vanishes if $t>t'$, and the $\theta(t-t')$ is redundant because the delta-function enforces $t=t'+|\vec{x}-\vec{x}'|>t'$ in the first term.

We now have the (retarded) solution to the wave equation for $h_{\alpha\beta}$ in the presence of the source $\bar{T}_{\alpha\beta}$:

$$h_{\alpha\beta}(x) = 4G \int d^4x' \, \frac{\bar{T}_{\alpha\beta}(x')}{|\vec{x}-\vec{x}'|} \, \delta(t-t'-|\vec{x}-\vec{x}'|)$$

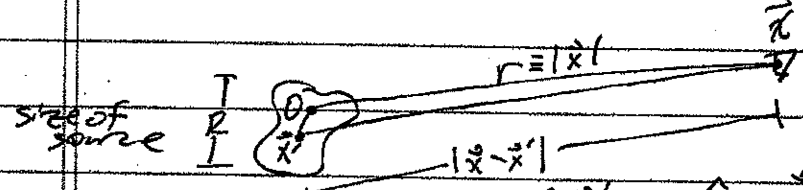
$$\boxed{h_{\alpha\beta}(x) = 4G \int d^3x' \, \frac{\bar{T}_{\alpha\beta}(\vec{x}', t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}}$$

Suppose $T_{\mu\nu}(\vec{x},t)$ describes an oscillating source,

$$T_{\mu\nu}(\vec{x},t) = T_{\mu\nu}(\vec{x},\omega)\, e^{-i\omega t} + \text{c.c.}$$

Then $h_{\alpha\beta}(x) = 4G \int d^3x' \, \bar{T}_{\alpha\beta}(\vec{x}',\omega) \exp(-i\omega t + i\omega|\vec{x}-\vec{x}'|)$
$+ \text{c.c.}$

The <u>radiation zone</u> is the region far from the source



$r \gg R$

$\approx r - \hat{x}\cdot\vec{x}'$, $\hat{x} \equiv \frac{\vec{x}}{r}$

$$h_{\alpha\beta}(x) \approx 4G \, \frac{1}{r} \exp(i\omega r - i\omega t) \int d^3x' \, \bar{T}_{\alpha\beta}(\vec{x}',\omega) \exp(-i\omega\hat{x}\cdot\vec{x}')$$
$+ \text{c.c.}$

This is like a plane wave with $\vec{k} = \omega\hat{x}$, $k^0 = \omega$

We can write

$$h_{\alpha\beta}(x) \approx \underbrace{\left(\frac{4G}{r} \int d^3x' \; \bar{T}_{\alpha\beta}(\vec{x}', \omega)\, e^{-i\vec{k}\cdot\vec{x}'} \right)}_{\epsilon_{\alpha\beta}(\vec{x},\omega)} e^{ik\cdot x} + c.c.$$

$\vec{k} = \omega\hat{x}$

In terms of $\tilde{T}_{\alpha\beta}(\vec{k},\omega) \equiv \int d^3x' \; T_{\alpha\beta}(\vec{x}',\omega)\, e^{-i\vec{k}\cdot\vec{x}'}$,

$$\boxed{h_{\alpha\beta} = \underbrace{\left(\frac{4G}{r} \left(\tilde{T}_{\alpha\beta}(\vec{k},\omega) - \frac{1}{2}\eta_{\alpha\beta}\tilde{T}(\vec{k},\omega) \right) \right)}_{\epsilon_{\alpha\beta}(\vec{x},\omega)} e^{ik\cdot x} + c.c.}$$

commentary 10/5

Quadrupolar Radiation

Suppose $\omega R \ll 1$ (nonrelativistic source)

where R = size of the source.

Then $\tilde{T}_{ij}(\vec{k},\omega) \approx \int d^3x \; T_{ij}(\vec{x},\omega)$ indep. of $\vec{k} = \omega\hat{x}$.

Conservation of angular momentum gives

$$\partial_\mu T^{\mu\nu} = \partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$\nu = j$: $\partial_0 T^{0j} + \partial_i T^{ij} = 0$ $\quad\|\quad$ $\nu = 0$: $\partial_0 T^{00} + \partial_i T^{i0} = 0$

∂_j: $\partial_0\partial_j T^{0j} + \partial_j\partial_i T^{ij} = 0$

$\quad -\partial_0\partial_0 T^{00} + \partial_j\partial_i T^{ij} = 0$

with $T^{\mu\nu}(\vec{x},t) = T^{\mu\nu}(\vec{x},\omega)\, e^{-i\omega t} + c.c.$,

$$\boxed{\partial_j\partial_i T^{ij} = -\omega^2 T^{00}}$$

Multiply by $x^k x^l$, integrate over $\vec{x}$:

$$\int d^3x \; x^k x^l \, \partial_i \partial_j T^{ij} = -\omega^2 \int d^3x \; T^{00}(\vec{x},\omega) \, x^k x^l$$

∂_j by parts $= -2 \int d^3x \; x^k \, \partial_i T^{il}$

∂_i by parts $= +2 \int d^3x \; T^{kl} \approx 2 \tilde{T}^{kl}(\vec{k},\omega)$

$$\Rightarrow \boxed{\tilde{T}^{kl}(\vec{k},\omega) \approx -\frac{\omega^2}{2} D^{kl}(\omega)}$$

where $D^{kl}(\omega)$ is $\frac{1}{3}\times$ the <u>Quadrupole Moment</u> of the energy density,

$$\boxed{D^{kl}(\omega) = \int d^3x \; T^{00}(\vec{x},\omega) \, x^k x^l}$$

Finally, we get the solution for $\bar{h}_{ij}$ in the radiation zone for nonrelativistic sources:

$$\boxed{\bar{h}_{ij} = -\frac{4G}{r}\frac{\omega^2}{2}\left(D_{ij}(\omega) e^{ik\cdot x}\right)} + \text{c.c.}$$

(where $\bar{h}_{ij} = h_{ij} - \frac{1}{2}\delta_{ij} h_\mu{}^\mu$)