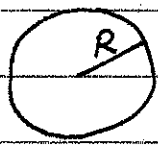
Zee VII.2

How to Make a Black Hole

Collapse of Pressureless Dust

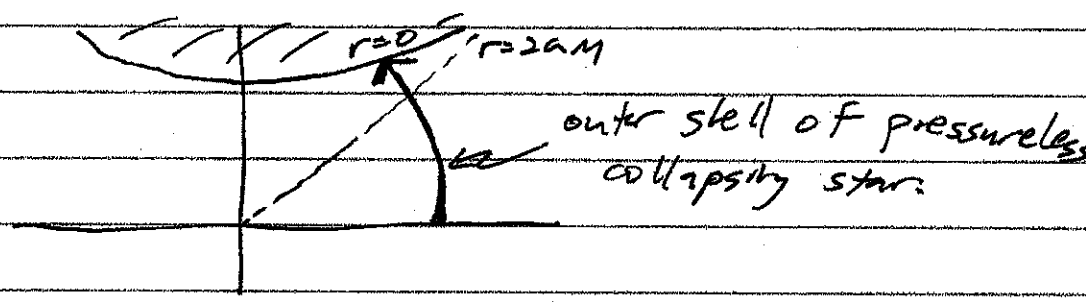


$$r > R: \; ds^2 = \frac{32\,(GM)^3\, e^{-r/2GM}}{r}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

Each dust particle on the surface (shell) follows a geodesic toward the center:

$$\begin{cases} r = \frac{R}{2}\left(1 + \cos\eta\right) \\ \tau = \frac{R}{2}\left(\frac{R}{2GM}\right)^{1/2}\left(\eta + \sin\eta\right) \end{cases}$$

Kruskal coordinates: $X^2 - T^2 = \left(\frac{r}{2GM} - 1\right) e^{r/2GM}$



Collapse happens in finite proper time τ, but light signals are delayed so outside observers do not see the shell cross the Schwarzschild radius.

Zee VII.4

Interior Solutions for Stars

Special Relativity: $T^{\alpha\beta} = \sum_n p_n^\alpha(t) \frac{dx_n^\beta}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$

$T^{00} = \sum_n E_n(t) \delta^3(\vec{x} - \vec{x}_n(t))$ Energy density

$T^{0i} = \sum_n E_n(t) \frac{dx_n^i}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$ Energy Flux in direction i

$T^{ji} = \sum_n p_n^j(t) \frac{dx_n^i}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$ Flux of j-component of momentum in direction i (stress)

Fluids: $N \to \infty$ limit

Perfect fluid: At each point there is a velocity $\vec{v}$ such that an observer moving w/ velocity $\vec{v}$ observes the fluid as isotropic (the same in every direction).

Local rest frame:
$T^{00} = \rho$ Energy density
$T^{0i} = 0$
$T^{ji} = p\,\delta^{ji}$ ← rotation-invariant
↖ pressure

In local rest frame, $U^\alpha = \frac{dx^\alpha}{d\tau} = (1,0,0,0)$ 4-velocity of fluid element

$$\boxed{T^{\alpha\beta} = p\,\eta^{\alpha\beta} + (p+\rho)U^\alpha U^\beta}$$

–tensor eqn. → valid in all frames.

Check: $T^{00} = -p + (p+\rho) = \rho$
$T^{0i} = 0$
$T^{ji} = p\,\delta^{ji}$ ✓

$$p_n^\mu = E_n \frac{dx_n^\mu}{dt} \longrightarrow T^{\alpha\beta} = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

Local rest frame: $T^{00} = \sum_n E_n \delta^3(\vec{x} - \vec{x}_n(t))$

$$p = \frac{1}{3}(T^{11} + T^{22} + T^{33}) = \frac{1}{3} \sum_n \frac{\vec{p}_n \cdot \vec{p}_n}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

Massive particles: $E > \frac{\vec{p}^{\,2}}{E} \Rightarrow \rho \geq 3p \geq 0$

Massless particles (photons): $E = \frac{\vec{p}^{\,2}}{E} \rightarrow \rho = 3p$

<u>General Relativity</u>: $T^{\alpha\beta} = p g^{\alpha\beta} + (p + \rho) U^\alpha U^\beta$

$$U^\alpha = \frac{dx^\alpha}{d\tau}, \quad g_{\alpha\beta} U^\alpha U^\beta = -1$$

<u>Stars</u>: Static, spherically symmetric solution to Einstein's equations

Look for solutions of the form

$$ds^2 = -e^{2\phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

Fluid elements are at rest $\Rightarrow U^\alpha = (e^{-\phi(r)}, 0, 0, 0)$

$$g_{\alpha\beta} U^\alpha U^\beta = -1 \quad \checkmark$$

$$\begin{cases} T^{\mu\nu} = g^{\mu\nu} p(r) + \big(p(r) + \rho(r)\big) U^\mu U^\nu \\ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu} \end{cases}$$

Define $m(r)$ by $e^{2\lambda(r)} \equiv \left(1 - \frac{2Gm(r)}{r}\right)^{-1}$

— like Schwarzschild

Weinberg 11.1 Einstein equations:

(1) $$\frac{dm}{dr} = 4\pi r^2 \rho$$

(2) $$\frac{dp}{dr} = -\frac{G\rho m}{r^2}\left(1+\frac{p}{\rho}\right)\left(1+\frac{4\pi p r^3}{m}\right)\left(1-\frac{2Gm}{r}\right)^{-1}$$

— Tolman-Oppenheimer-Volkov eqn. of hydrostatic equilibrium

(3) $$\frac{d\phi}{dr} = \frac{Gm}{r^2}\left(1+\frac{4\pi p r^3}{m}\right)\left(1-\frac{2Gm}{r}\right)^{-1}$$

Solutions:

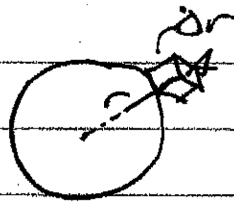
$$(1) \Rightarrow m(r) = 4\pi \int_0^r \rho(r')\, r'^2 dr' + \cancel{\text{const.}}$$

Good behavior at $r=0$: $e^{2\lambda(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1} \Rightarrow m(0)=0$

Assume $2Gm(r) < r$ for all r.

(2): To understand the meaning of the Tolman-Oppenheimer-Volkov equation, consider the Newtonian limit, $\begin{cases} p \ll \rho \\ Gm(r) \ll r \end{cases}$

$$\Rightarrow \frac{dp}{dr} \approx -\frac{G\rho m(r)}{r^2}$$ — Equation of hydrostatic equilibrium



Gravitational force on infinitesimal box:

$$F_g = \frac{G\left(A\Delta r\, \rho(r)\right) m(r)}{r^2}$$

$$= -\left(p(r+\Delta r)A - p(r)A\right)$$

Divide by Δr: $-\frac{dp}{dr} = \frac{G\rho\, m(r)}{r^2}$ hydrostatic equilibrium

(3) To understand the meaning of eq. (3), consider the Newtonian limit again.

$$\frac{d\phi}{dr} \approx \frac{Gm(r)}{r^2} \rightarrow \phi(r) = G\int_0^r \frac{m(r)}{r^2}\, dr$$

= Newtonian potential

$$e^{2\phi(r)} \approx 1 + 2\phi(r)$$

Not necessarily in the Newtonian limit:

At the edge of the star $p = 0$

$$r > R: \; p = \rho = 0 \rightarrow \frac{d\phi}{dr} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}$$

$$\rightarrow e^{2\phi} = 1 - \frac{2GM}{r}$$ Schwarzschild solution

To model a star: (1) Assume an equation of state $p(\rho)$.

(2) Assume a central value for ρ, i.e. $\rho(0) = \rho_c \rightarrow p(0) = p(\rho_c)$

(3) Integrate Einstein eqs 1+2 to larger r, s.t. $m(0) = 0$.

(4) End integration at r s.t. $p(r) = 0 \rightarrow r = R$.

(5) Integrate Einstein eqn. 3 to find $\phi(r)$.

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Example: Incompressible Fluid

$\rho =$ constant, indep. of p

$$\rho(r) = \begin{cases} \rho_0 & r \le R \\ 0 & r > R \end{cases}$$

$$m(r) = \frac{4}{3}\pi \rho_0 r^3$$

$$\frac{dp}{dr} = -G\frac{4}{3}\pi\rho_0^2 r\left(1+\frac{p}{\rho_0}\right)\left(1+\frac{3p}{\rho_0}\right)\left(1-\frac{8}{3}\pi\rho_0 G r^2\right)^{-1}$$

Integrate:

$$p(r) = \rho_0 \frac{\left[\left(1-\frac{2GM}{R}\right)^{1/2} - \left(1-\frac{2GMr^2}{R^3}\right)^{1/2}\right]}{\left(1-\frac{2GMr^2}{R^3}\right)^{1/2} - 3\left(1-\frac{2GM}{R}\right)^{1/2}}$$

$p(R) = 0 \rightarrow$ defines R (edge of star)

$r = 0$: Denominator of $p(r)$ would vanish if

$$1 - 3\left(1-\frac{2GM}{R}\right)^{1/2} = 0$$

$$\rightarrow \frac{GM}{R} = 4/9$$

If $\frac{GM}{R} > \frac{4}{9}$ then $p(r) \to \infty$ for some $r > 0$
— unphysical

$\rightarrow$ $\boxed{GM < \frac{4R}{9}}$ Bound on mass of star

In fact, the bound $GM < \frac{4R}{9}$ follows more generally as long as $\rho(r) > 0$ and $\frac{d\rho}{dr} < 0$.

wenlbey 11.3 White Dwarfs. - Old star burns its fuel, cools and contracts.

Low temperature $T \rightarrow$ electrons freeze into lowest available energy levels.

Number of electrons / unit volume:

$$n = \underbrace{2}_{\text{spins}} \cdot \underbrace{\frac{4\pi}{(2\pi\hbar)^3}}_{\text{from density of states}} \int_0^{k_F} k^2 \, dk \quad \leftarrow \text{Fermi momentum}/\hbar$$

$$\Rightarrow \boxed{n = \frac{k_F^3}{3\pi^2\hbar^3}}$$

Mass density mainly from nucleons:

$$\rho = n \underbrace{m_N}_{\text{avg. nucleon mass}} \underbrace{\mu}_{\text{\# nucleons/electron}}$$

Iron: $\mu = \frac{56}{26}$

$$\Rightarrow \boxed{k_F = \hbar\left(\frac{3\pi^2\rho}{m_N\mu}\right)^{1/3}}$$

Electron pressure:

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F} \frac{k^2}{\sqrt{k^2+m_e^2}}\, k^2\,dk$$

$\uparrow \frac{p^2}{E}$ (from energy-momentum tensor T^{ij})

If $k_F \gg m_e$:

$$p \approx \frac{8\pi\, k_F^4}{12(2\pi\hbar)^3} = \frac{\hbar}{12\pi^2}\left(\frac{3\pi^2}{m_N\mu}\right)^{4/3}\rho^{4/3}$$

$$\equiv K\rho^{4/3} \quad \text{Equation of state}$$

Weinberg 11.3

1) Integrating the equation for hydrostatic equilibrium with this equation of state gives $\rho(r)$.
2) pressure vanishes $\rightarrow$ radius of star R.
3) $M = \int_0^R 4\pi r'^2 \rho(r')\,dr'$

$$\Rightarrow M = \tfrac{1}{2}(3\pi)^{1/2}(2.01824)\left(\frac{\hbar^{3/2}c^{3/2}}{G^{3/2}m_N^2\mu^2}\right)$$

$$= \boxed{5.87\,\mu^{-2}M_\odot \equiv M_C}$$

For smaller k_F, $M < M_C$. Hence, white dwarf stars have mass $< M_C \equiv$ Chandrasekhar limit.

In fact, for $k_F \simeq 5 m_e$, it is energetically favorable for electrons to be captured by nuclei, converting protons to neutrons
$\Rightarrow M < 1.2 M_\odot$, close to Chandrasekhar limit w/ $\mu = \frac{56}{26}$