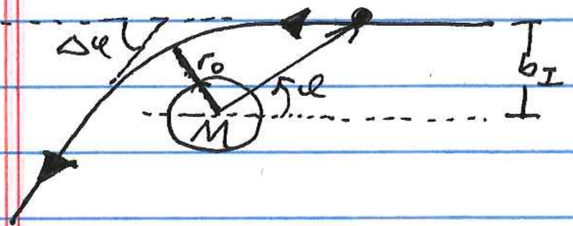
Bending of Light by a Massive Object



r_0 = value of r at closest approach $(t=0)$

b_I = impact parameter

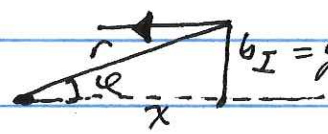
$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)\frac{\tilde{L}^2}{r^2} = \tilde{E}^2$$

$$r^2 \frac{d\varphi}{d\tau} = \tilde{L}$$

$$\Rightarrow \frac{d\varphi}{dr} = \pm \frac{\tilde{L}}{r^2}\left[\tilde{E}^2 - \left(1 - \frac{2GM}{r}\right)\frac{\tilde{L}^2}{r^2}\right]^{-1/2}$$

Define $b = \frac{\tilde{L}}{\tilde{E}} = \left(r^2 \frac{d\varphi}{d\tau}\right) \Big/ \left(\frac{dt}{d\tau}\right)$

Early times:



$y = b_I$

$x \approx -t = r\cos\varphi$

$r \approx |t|$

$$\tan\varphi = \frac{b_I}{x} \approx -\frac{b_I}{t}$$

$$\frac{d}{dt}: \quad \sec^2\varphi \frac{d\varphi}{dt} = \frac{b_I}{t^2}, \quad \sec^2\varphi \, t^2 = r^2$$

$$\Rightarrow r^2 \frac{d\varphi}{dt} = b_I$$

$$\boxed{b = \frac{\tilde{L}}{\tilde{E}} = r^2 \frac{d\varphi}{dt} = b_I}$$

$$\Rightarrow \boxed{\frac{d\varphi}{dr} = \pm\left[\frac{r^4}{b^2} - (r - 2GM)r\right]^{-1/2}}$$

Change in φ along trajectory:

$$\varphi_f - \varphi_i = \int d\varphi = 2\int_{r_0}^{\infty} dr \left[\frac{r^4}{b^2} - r(r-2GM)\right]^{-1/2}$$

At closest approach $r = r_0$, $\frac{dr}{d\tau} = 0$

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)\frac{\tilde{L}^2}{r^2} = E^2$$

$$\left(1 - \frac{2GM}{r_0}\right)\frac{\tilde{L}^2}{r_0^2} = \tilde{E}^2$$

$$\frac{1}{r_0^2}\left(1 - \frac{2GM}{r_0}\right) = \frac{\tilde{E}^2}{\tilde{L}^2} = \frac{1}{b^2}$$

$$\varphi_f - \varphi_i = 2\int_{r_0}^{\infty} dr \left[\frac{r^4}{r_0^2}\left(1 - \frac{2GM}{r_0}\right) - r^2 + 2GMr\right]^{-1/2}$$

Let $u = 1/r$:

$$\varphi_f - \varphi_i = 2\int_0^{1/r_0} du \left[\left(1 - \frac{2GM}{r_0}\right)\frac{1}{r_0^2} - u^2 + 2GMu^3\right]^{-1/2}$$

If $M = 0$, $\varphi_f - \varphi_i = 2\int_0^{1/r_0} du \left[\frac{1}{r_0^2} - u^2\right]^{-1/2} = \pi$

(Exercise)

If $M \neq 0$, but $\frac{2GM}{r_0} \ll 1$:

$$\Delta\varphi = (\varphi_f - \varphi_i) - \pi$$

$$= 2\int_0^{1/r_0} du \left[\left(\frac{1}{r_0^2} - u^2\right) - 2GM\left(\frac{1}{r_0^3} - u^3\right)\right]^{-1/2} - \pi$$

$$\approx \cancel{2\int_0^{1/r_0} du \left[\frac{1}{r_0^2} - u^2\right]^{-1/2}} + 2GM\int_0^{1/r_0} du \frac{\left(\frac{1}{r_0^3} - u^3\right)}{\left(\frac{1}{r_0^2} - u^2\right)^{3/2}} \cancel{-\pi}$$

$$\Delta\varphi \simeq 2GM\int_0^{1/r_0} \frac{du}{\left(\frac{1}{r_0^2}-u^2\right)^{3/2}}\left[\left(\frac{1}{r_0^2}-u^2\right)\left(\frac{1}{r_0}+u\right)+\frac{u^2}{r_0}-\frac{u}{r_0^2}\right]$$

$$= 2GM\int_0^{1/r_0} du\left\{\frac{\left(\frac{1}{r_0}+u\right)}{\left(\frac{1}{r_0^2}-u^2\right)^{1/2}}+\frac{\left(\frac{u^2}{r_0}-\frac{u}{r_0^2}\right)}{\left(\frac{1}{r_0^2}-u^2\right)^{3/2}}\right\}$$

Let $u = \frac{1}{r_0}\sin\theta$, $\quad du = \frac{1}{r_0}\cos\theta\, d\theta$

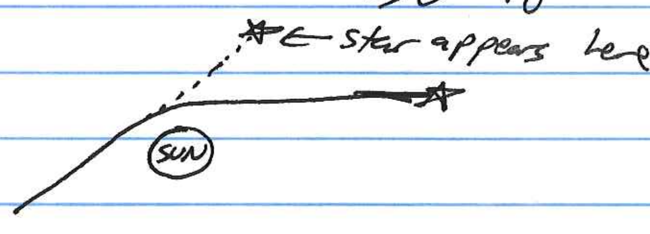
$$\frac{1}{r_0^2}-u^2 = \frac{1}{r_0^2}\cos^2\theta$$

$$\Delta\varphi \simeq \frac{2GM}{r_0}\int_0^{\pi/2} d\theta\left\{\frac{\cancel{\cos\theta}\,(1+\sin\theta)}{\cancel{\cos\theta}}+\frac{\cos\theta\left(\sin^2\theta-\sin\theta\right)}{(\cos\theta)^3}\right\}$$

$$= \frac{2GM}{r_0}\int_0^{\pi/2} d\theta\left\{\sin\theta+\frac{(1-\sin\theta)}{\cos^2\theta}\right\}$$

$$= \frac{2GM}{r_0}(-\cos\theta)\Big|_0^{\pi/2}+\frac{2GM}{r_0}\,\frac{(\sin\theta-1)}{\cos\theta}\Big|_0^{\pi/2}$$

$$\boxed{\Delta\varphi \simeq \frac{4GM}{r_0}}$$



Deflection of starlight by sun:

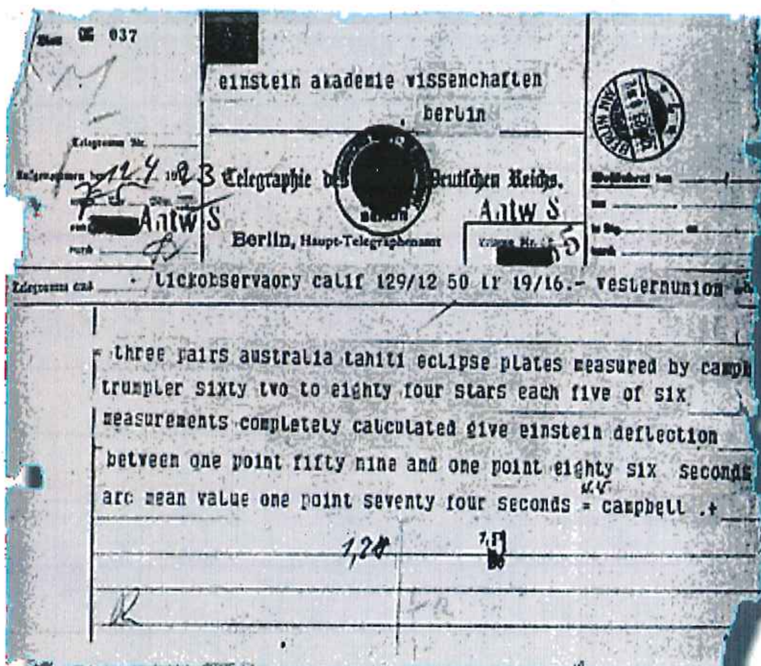
$$r_0 = R_\odot = 6.96\times10^5 \text{ km}$$

$$GM_\odot = 1.48 \text{ km}$$

$$\Rightarrow \Delta\varphi = \left(8.5\times10^{-6}\text{ rad}\right)\times\frac{360^\circ}{2\pi}\cdot\frac{3600''}{1^\circ}$$

$$= 1.75'' \text{ arc deflection.}$$

In 1919, Arthur Stanley Eddington led an expedition to the island of Principe in the Gulf of Guinea to photograph stars in the vicinity of the sun during the May 29 solar eclipse. A second expedition led by Andrew Crommelin went to Sobral, in Brazil. Their observations confirmed the magnitude of the bending of light predicted by Einstein, and propelled Einstein to super-famedom.



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einstein akademie vissenchaften
berlin

Telegramm Nr.
Aufgenommen den 12.4. 1923 Telegraphie des Deutschen Reichs.
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Berlin, Haupt-Telegraphenamt
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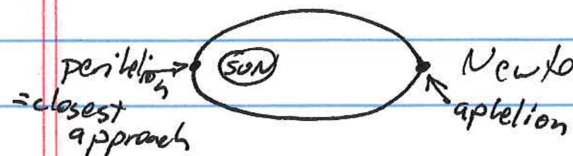
three pairs australia tahiti eclipse plates measured by camp
trumpler sixty two to eighty four stars each five of six
measurements completely calculated give einstein deflection
between one point fifty nine and one point eighty six seconds
arc mean value one point seventy four seconds = campbell .+

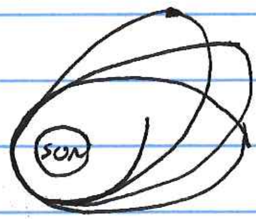
1,24

Eddington → Einstein Telegram

Precession of the Perihelion of Planetary Orbits



Newtonian — closed elliptical orbits



GR — Orbit precesses

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

$$r^2 \frac{d\varphi}{d\tau} = \tilde{L}$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left[\tilde{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)\right]\frac{r^4}{\tilde{L}^2}$$

$u = \frac{1}{r}$:

$$\boxed{\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{\tilde{L}^2}\left[\tilde{E}^2 - (1 - 2GMu)(1 + \tilde{L}^2 u^2)\right]}$$

Analogous Newtonian problem: ignore u^3 term.

$$\left(\frac{du}{d\varphi}\right)^2 = -a_0 + a_1 u - u^2$$, where a_0, a_1 depend on $\tilde{E}, \tilde{L}, GM$.

Let $u = y + \frac{a_1}{2}$ (complete the square)

$$\left(\frac{dy}{d\varphi}\right)^2 = \left(-a_0 + \frac{a_1^2}{2} - \frac{a_1^2}{4}\right) - y^2 = \left(-a_0 + \frac{a_1^2}{4}\right) - y^2$$

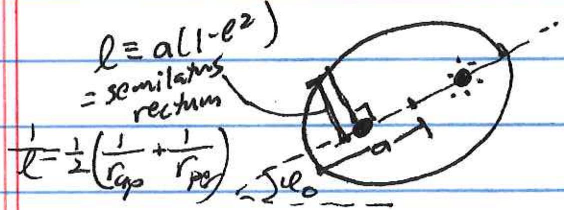
Solution: $y = \left(-a_0 + \frac{a_1^2}{4}\right)^{1/2} \cos(\varphi - \varphi_0)$

$$u = \frac{a_1}{2} + \left(-a_0 + \frac{a_1^2}{4}\right)^{1/2} \cos(\varphi - \varphi_0)$$

$$\frac{1}{r} = \frac{a_1}{2} + \left(-a_0 + \frac{a_1^2}{4}\right)^{1/2} \cos(\varphi - \varphi_0)$$

$$\boxed{\frac{1}{r} = \frac{a_1}{2}\left[1 + \left(-\frac{4a_0}{a_1^2} + 1\right)^{1/2} \cos(\varphi - \varphi_0)\right]}$$

This is an ellipse, $r = \dfrac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)}$



e = eccentricity $= \dfrac{r_{ap} - r_{per}}{r_{ap} + r_{per}}$

a = semimajor axis $= \dfrac{r_{ap} + r_{per}}{2}$

$$r_{ap} = (1+e)a, \quad r_{per} = (1-e)a$$

In our case, $e = \left(1 - \frac{4a_0}{a_1^2}\right)^{1/2}$

$$a = \frac{a_1}{2a_0}$$

$$l = \frac{a_1}{2a_0}\left(\frac{4a_0}{a_1^2}\right) = \frac{2}{a_1}$$

Correction to orbit due to GR due to u^3 term:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{\tilde{E}^2-1}{\tilde{L}^2} + \frac{2GM}{\tilde{L}^2}u - u^2 + 2GMu^3$$

$$= -a_0 + a_1 u - u^2 + a_3 u^3$$

$$a_0 = \frac{1-\tilde{E}^2}{\tilde{L}^2}, \quad a_1 = \frac{2GM}{\tilde{L}^2}, \quad a_3 = 2GM$$

Let $u = u_0 + y$, choose u_0 to eliminate linear term in y

$$\rightarrow a_1 - 2u_0 + 3a_3 u_0^2 = 0$$

$$\left(\frac{dy}{d\varphi}\right)^2 = \left[-a_0 + a_1 u_0 - u_0^2 + a_3 u_0^3\right] + \left[-1 + 3a_3 u_0\right] y^2 + a_3 y^3$$

Circular orbit: $y = 0$; $r = \frac{1}{u_0} = \text{const.}$

– solution if $-a_0 + a_1 u_0 - u_0^2 + a_3 u_0^3 = 0$

Nearly circular orbit: y small $\rightarrow$ neglect y^3 piece.

$$\left(\frac{dy}{d\varphi}\right)^2 \approx (\text{const}) - (1 - 3a_3 u_0) y^2$$

solution: $y = (\text{const})^{1/2} \cos\left(\sqrt{1-3a_3u_0}\,(\varphi - \varphi_0)\right)$

$$= (\text{const})^{1/2} \cos\left(\sqrt{1 - \frac{6GM}{r_0}}\,(\varphi - \varphi_0)\right)$$

$r_0 = \frac{1}{u_0} \approx \frac{2}{a_1}$ from Newtonian approximation.

$= \ell$ (semilatus rectum)

$\frac{1}{\ell} = \frac{1}{2}\left(\frac{1}{r_{ap}} + \frac{1}{r_{per}}\right) \approx \frac{1}{r_{ap}}$ for nearly circular orbit

Perihelion = closest approach
$\rightarrow r$ minimum $\rightarrow y$ maximum

$$\cos\left(\sqrt{1-\frac{6GM}{r_0}}\,(\varphi-\varphi_0)\right) \text{ maximum}$$

$$\rightarrow \varphi-\varphi_0 = 0,\ \frac{2\pi}{\sqrt{1-\frac{6GM}{r_0}}} \approx 2\pi\left(1+\frac{3GM}{r_0}\right), \text{ etc.}$$

r returns to r_0 if $\varphi \rightarrow \varphi + 2\pi + \frac{6\pi GM}{r_0}$

Precession of perihelion: $\boxed{\Delta\varphi = \frac{6\pi GM}{r_0} \ \frac{\text{radians}}{\text{revolution}}}$

Mercury: $r_0 = 5.55\times 10^7$ km
$GM_\odot = 1.47$ km
$\rightarrow \Delta\varphi = 4.99\times 10^{-7}$ rad/revolution
1 revolution = 0.24 years

$$\rightarrow \boxed{\Delta\varphi = 43.0'' \text{ arc/century}}$$ (Exercise)

Newtonian perturbations (due to Jupiter, etc.):
$\Delta\varphi_N = 5557.62 \pm 0.20$ "/century
Observed: $\Delta\varphi_{OBS} = 5600.73 \pm 0.41$ "/century

GR contribution: $\Delta\varphi_{OBS} - \Delta\varphi_N = 43.1$ "/century
– agrees w/ prediction!

Radar Echo Delay

It is worth mentioning another test of GR based on gravitational time delay.

A light signal is reflected off of another planet, and the time of return is measured. GR predicts the time delay of the signal as a function of sun's mass, Earth's radius, planet's radius, distance at closest approach,

(cf. Weinberg 8.7)

— Shapiro 1968, 1971

Targets other than planets may also be used, e.g. artificial satellites which reemit signals (Anderson 1975), anchored spacecraft in orbit around planet or landed on planet (Anderson, Shapiro, Rosenberg 1979)