The Schwarzschild Metric

Weinberg 8.1–8.2

Zee VII.1

Consider a spacetime metric of the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

(static, isotropic)

i.e.

$$g_{rr} = A(r), \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2\sin^2\theta, \quad g_{tt} = -B(r)$$

$$g^{rr} = A^{-1}(r), \quad g^{\theta\theta} = r^{-2}, \quad g^{\varphi\varphi} = r^{-2}(\sin\theta)^{-2}, \quad g^{tt} = -B^{-1}(r)$$

We calculate the affine connection:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right)$$

Nonvanishing components:

$$\Gamma^r_{rr} = \frac{1}{2}A^{-1}(r)\left(A'(r) + A'(r) - A'(r)\right) = \frac{1}{2}A^{-1}(r)\,A'(r)$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}A^{-1}(r)\left(-\frac{d}{dr}(r^2)\right) = -A^{-1}(r)\,r$$

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2}A^{-1}(r)\left(-\frac{\partial}{\partial r}(r^2\sin^2\theta)\right) = -A^{-1}(r)\,r\sin^2\theta$$

$$\Gamma^r_{tt} = \frac{1}{2}A^{-1}(r)\,B'(r), \qquad \Gamma^t_{tr} = \Gamma^t_{rt} = -\frac{1}{2}B^{-1}(r)\,(-B'(r))$$
$$= \frac{1}{2}B^{-1}(r)\,B'(r)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2}r^{-2}\frac{d}{dr}(r^2) = \frac{1}{r}$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2}r^{-2}\left(-\frac{\partial}{\partial\theta}(r^2\sin^2\theta)\right) = -\sin\theta\cos\theta$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{2}(r^2\sin^2\theta)^{-1}\,\partial_r(r^2\sin^2\theta) = r^{-1}$$

$$\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \frac{1}{2}(r^2\sin^2\theta)^{-1}\frac{\partial}{\partial\theta}(r^2\sin^2\theta) = \cot\theta$$

The Ricci tensor is given by

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\lambda\eta}$$

$$R_{rr} = \frac{d}{dr}\left(\cancel{\frac{A'(r)}{2A(r)}} + \frac{B'(r)}{2B(r)} + \cancel{\frac{1}{r} + \frac{1}{r}}\right) - \frac{d}{dr}\left(\cancel{\frac{A'(r)}{2A(r)}}\right)$$

$$+ \left(\cancel{\left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2} + \left(\frac{A'}{2A}\right)^2 + \left(\frac{B'}{2B}\right)^2\right)$$

$$- \left(\frac{A'}{2A}\right)\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{1}{r} + \frac{1}{r}\right)$$

$$= \frac{B''}{2B} - \frac{1}{2B^2}(B')^2 + \cancel{\frac{1}{4A^2}(A')^2} + \frac{1}{4B^2}(B')^2$$

$$- \cancel{\frac{1}{4A^2}(A')^2} - \frac{A'B'}{4AB} - \frac{A'}{rA}$$

$$= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\left(\frac{A'}{A}\right)$$

Similarly,

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}$$

$$R_{\varphi\varphi} = \sin^2\theta\left(-1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}\right)$$

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4}\left(\frac{B'}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\left(\frac{B'}{A}\right)$$

$R_{\mu\nu} = 0$ for $\mu \neq \nu$.

The vacuum Einstein eqs are $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.

Contracting μ, ν: $R - \frac{1}{2} \cdot 4R = -R = 0$

$$\Rightarrow \boxed{R_{\mu\nu} = 0} \quad \text{in vacuum.}$$

Use $\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$

$$\Rightarrow \frac{B'}{B} = -\frac{A'}{A} \Longrightarrow A(r)\,B(r) = \text{const.}$$

As $r \to \infty$ we will insist that the metric approach the Minkowski metric in spherical coordinates,

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

$$\rightarrow \boxed{A(r) = 1/B(r)}$$

$$R_{\theta\theta} = -1 + \frac{r}{2} B\left(-B \frac{d}{dr}(1/B) + B'/B\right) + B = 0$$

$$= -1 + rB' + B = 0$$

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\left(\frac{B'}{B}\right)\left(\cancel{B \frac{d}{dr}(1/B) + B'/B}\right) - \frac{1}{r}\left(B \frac{d}{dr}(1/B)\right)$$

$$= \frac{B''}{2B} + \frac{B'}{rB} = \frac{1}{2rB} \frac{d}{dr}(R_{\theta\theta})$$

So, if $R_{\theta\theta} = 0$ everywhere, then $R_{rr} = 0$ everywhere.

$$R_{\theta\theta} = \frac{d}{dr}(r B(r)) - 1 = 0$$

$$\Longrightarrow r B(r) = r + \text{const.}$$

To fix the constant, we will assume that at large r, the metric agrees with the Newtonian limit,

$$g_{tt} = -B \xrightarrow[r\to\infty]{} -1-2\phi \,, \text{ where}$$

$(G \equiv G_N)$

$\phi = -\frac{GM}{r}$ is the potential due to a point mass M at $r=0$. (But note that there is no mass sitting at $r=0$ in the Schwarzschild spacetime.)

Hence, $B(r) = 1 - \frac{2GM}{r}$

$$A(r) = \left(1 - \frac{2GM}{r}\right)^{-1}$$

$$\boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$

Schwarzschild Metric (1916) in Standard Form.

Defining a new radius variable

$$r \equiv \rho\left(1+\frac{GM}{2\rho}\right)^2,$$

Exercise.

$$\boxed{ds^2 = \frac{-(1-GM/2\rho)^2}{(1+GM/2\rho)^2}dt^2 + \left(1+\frac{GM}{2\rho}\right)^4\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)}$$

Schwarzschild metric in Isotropic Form.

We can calculate the total energy and momentum in any system whose metric outside some region is given by the Schwarzschild metric.

Weinberg 7.6

Write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The energy-momentum tensor of matter + gravitation is

$$-8\pi G_N T^{\mu\nu} = R^{(1)\mu\nu} - \frac{1}{2}\eta^{\mu\nu} R^{(1)}$$
$$= \frac{1}{2}\left(\partial^\mu \partial^\nu h^\lambda{}_\lambda - \partial_\lambda \partial^\nu h^{\lambda\mu} - \partial_\lambda \partial^\mu h^{\lambda\nu} + \partial_\lambda \partial^\lambda h^{\mu\nu} - \eta^{\mu\nu} \partial_\lambda \partial^\lambda h^\rho{}_\rho + \eta^{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho}\right)$$

where indices here are contracted with $\eta_{\alpha\beta}$. This can be written as a divergence:

$$\boxed{-8\pi G_N T^{\mu\nu} = \partial_\rho Q^{\rho\mu\nu}}, \text{ where}$$

$$\boxed{Q^{\rho\nu\lambda} = \frac{1}{2}\left\{\partial^\nu h^\mu{}_\mu \eta^{\rho\lambda} - \partial^\rho h^\mu{}_\mu \eta^{\nu\lambda} - \partial_\mu h^{\mu\nu}\eta^{\rho\lambda} + \partial^\mu h_\mu{}^\rho \eta^{\nu\lambda} + \partial^\rho h^{\nu\lambda} - \partial^\nu h^{\rho\lambda}\right\}}$$

Note that $Q^{\rho\nu\lambda} = -Q^{\nu\rho\lambda}$. (Consider the linearized Bianchi identity $\partial_\mu T^{\mu\nu} = 0$.)

Then $P^\lambda = -\frac{1}{8\pi G_N}\int_V \partial_\rho Q^{\rho 0\lambda}\, d^3x = -\frac{1}{8\pi G_N}\int \partial_i Q^{i0\lambda}\, d^3x$

$$\overset{\text{Gauss' theorem}}{=} -\frac{1}{8\pi G_N}\int Q^{i0\lambda} n_i r^2 d\Omega, \text{ where}$$

$r \equiv \sqrt{x^i x^i}$, $n_i \equiv x^i/r$, $d\Omega = \sin\theta\, d\theta\, d\varphi$

The total energy is

$$P^0 = -\frac{1}{8\pi G_N}\int Q^{i00} n^i r^2 d\Omega$$

$$= -\frac{1}{16\pi G_N}\int\left\{-\frac{\partial}{\partial x^i} h^\mu{}_\mu (-1) + \frac{\partial}{\partial x^\mu} h^{\mu i}(-1) + \frac{\partial}{\partial x^i} h^{00}\right\} n^i r^2 d\Omega$$

$$\boxed{P^0 = -\frac{1}{16\pi G_N}\int\left\{\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j}\right\} n^i r^2 d\Omega}$$

$$P^j = -\frac{1}{8\pi G_N}\int Q^{i0j} n^i r^2 d\Omega$$

$$\boxed{P^j = -\frac{1}{16\pi G_N}\int\left\{-\frac{\partial}{\partial t} h_{kk}\delta_{ij} + \frac{\partial}{\partial x^k} h_{k0}\delta_{ij} - \frac{\partial}{\partial x^i} h_{j0} + \frac{\partial}{\partial t} h_{ij}\right\} n^i r^2 d\Omega}$$

For the Schwarzschild metric, $h_{k0} = 0$ and $\frac{\partial}{\partial t} h_{ij} = 0$, so $P^j = 0$. This is not surprising: the static, isotropic (same in every direction) Schwarzschild solution has no spatial momentum.

It is convenient to define quasi-Minkowskian coordinates:

$x^1 \equiv r\sin\theta\cos\varphi$, $x^2 \equiv r\sin\theta\sin\varphi$, $x^3 \equiv r\cos\theta$

where r, θ, φ are the standard Schwarzschild coords. The metric becomes,

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left\{\left(1 - \frac{2GM}{r}\right)^{-1} - 1\right\} r^{-2}\left(x^i dx^i\right)^2 + dx^i dx^i}$$

where $r = \sqrt{x^i x^i}$, $dr = \frac{1}{r} x^i dx^i$

$$h_{ij} \equiv g_{ij} - \delta_{ij} \xrightarrow[r\to\infty]{} \frac{2GM}{r} n_i n_j + \mathcal{O}(1/r^2)$$

where $n_i \equiv x^i/r$, $n_i n_i = 1$

Use $\frac{\partial r}{\partial x^i} = \frac{1}{r} x^i = n_i$

$$\frac{\partial n_i}{\partial x^j} = \frac{1}{r}\delta_{ij} - \frac{1}{r^2} x^i x^j = \frac{1}{r}(\delta_{ij} - n_i n_j)$$

$$\Rightarrow \frac{\partial}{\partial x^i} h_{jj} - \frac{\partial}{\partial x^j} h_{ij} \xrightarrow[r\to\infty]{} \frac{\partial}{\partial x^i}\left(\frac{2GM}{r}\right) - \frac{\partial}{\partial x^j}\left(\frac{2GM}{r} n_i n_j\right)$$

$$= -\frac{2GM}{r^2} n_i - \left\{ -\frac{2GM}{r^2} n_j n_i n_j + \frac{2GM}{r^2}(\delta_{ij} - n_i n_j) n_j + \frac{2GM}{r^2} n_i \underbrace{(\delta_j{}^j - n_j n_j)}_{2} \right\}$$

$$= -\frac{4GM}{r^2} n_i$$

$$\Rightarrow P^0 = -\frac{1}{16\pi G}\int\left(-\frac{4GM}{r^2}\right) n_i n_i \, r^2 \, d\Omega$$

$$\boxed{P^0 = M}$$

Hence, the total energy of matter + gravitation in any spacetime which is described by the Schwarzschild metric outside some region is M.

The Schwarzschild Horizon

The Schwarzschild metric is singular at $r_s = 2GM$, where $g_{tt} = 0$, $g_{rr} = \infty$.
The gravitational redshift and time dilation become divergent as $r \to 2GM$. The value $r_s = 2GM$ is called the Schwarzschild radius, and the surface $r = 2GM$ is called the Schwarzschild horizon.

Light travels along <u>null geodesics</u> with $ds^2 = 0$.
A trajectory with $\theta, \varphi = \text{const.}$ satisfies

$$\frac{dr}{dt} = \left(1 - \frac{2GM}{r}\right) \to 0 \quad \text{as} \quad r \to 2GM.$$

The Schwarzschild horizon separates causally distinct regions in the spacetime, $r < 2GM$ and $r > 2GM$.
$\Rightarrow$ Schwarzschild is a <u>black hole</u> spacetime.
Note that the Schwarzschild solution is only relevant where there is no matter, so $T_{\mu\nu} = 0$. If matter is not compressed within the Schwarzschild radius then the geometry will be regular.

Typical Schwarzschild radii:

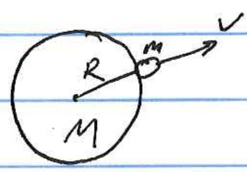
Sun: $2GM_\odot = 2.95$ km

proton: $2GM_p = 10^{-52}$ m

Earth: $2GM_E = 8.87 \times 10^{-3}$ m

It is important to note that despite the physical importance of the Schwarzschild horizon, the spacetime curvature $R_{\mu\nu\rho\sigma}$ is nonsingular at $r=2GM$. An observer falling through the horizon will feel no infinite stresses at the horizon.

It is amusing to compare the Schwarzschild radius to the size of an object whose Newtonian escape velocity is the speed of light c.



$$\frac{1}{2} m V_{escape}^2 - \frac{GMm}{R} = 0$$

$$V_{escape}^2 = c^2 = \frac{2GM}{R}$$

$$R = \frac{2GM}{c^2} = R_s$$

The Schwarzschild geometry approximates the spacetime in the neighborhood of the sun, so we will analyze trajectories which describe planetary motion and bending of light by the sun.