Physics 786, Spring 2012
Problem Set 3 Due Wednesday, February 22, 2012.

1. *Gravitational Waves*

a) Suppose that a gravitational plane wave has wavevector $k^\mu = (k, 0, 0, k)$ and polarization tensor $\epsilon_{\mu\nu}$ satisfying the harmonic gauge condition,

$$k^\mu \epsilon_{\mu\nu} = \tfrac{1}{2} k_\nu \epsilon_\mu{}^\mu .$$

Show that the components of $\epsilon_{\mu\nu}$ are related as follows:

$$\begin{aligned} &\epsilon_{01} = -\epsilon_{31}, \quad \epsilon_{02} = -\epsilon_{32}, \quad \epsilon_{22} = -\epsilon_{11}, \\ &\epsilon_{03} = -\tfrac{1}{2}\left(\epsilon_{33} + \epsilon_{00}\right). \end{aligned}$$

b) Show that by making a gauge transformation which preserves the harmonic gauge condition, $\epsilon_{\mu\nu}$ from part (a) can be put in the form,

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{xx} & \epsilon_{xy} & 0 \\ 0 & \epsilon_{xy} & -\epsilon_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. *Particle motion in a gravitational field*

Derive the equations of motion for a particle's trajectory in a gravitational field described by $g_{\mu\nu}(x)$, beginning with the action

$$S = \frac{m}{2} \int d\tau \, \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} g_{\mu\nu},$$

and show that they are equivalent to the equations describing the motion a freely-falling particle in coordinates with spacetime metric $g_{\mu\nu}$.

3. *Particle motion in scalar gravity*

Suppose that gravitation were described by a scalar field $\phi(x)$ which coupled to the trace of the energy-momentum tensor, so that the particle action had the form,

$$S = m \int d\tau \, \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu} - \lambda \int d^4x \, \phi \, T_\mu{}^\mu ,$$

where λ is a coupling constant and $T^{\mu\nu}$ is the energy-momentum tensor of the particle,

$$T^{\mu\nu} = m\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta^3(\mathbf{x} - \mathbf{x}(t))\frac{d\tau}{dt}.$$

For a prescribed field $\phi(x)$ (*i.e.* ignoring the backreaction of the particle motion on $\phi(x)$), derive the equations of motion for the particle's trajectory and show that those equations can be written in the form,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0,$$

for some $\Gamma^\mu_{\alpha\beta}$ which depends on $\phi(x)$.

Can you define a $g_{\mu\nu}$ in terms of $\phi(x)$ such that $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols in the spacetime metric given by $g_{\mu\nu}$?