Physics 786, Spring 2012
Problem Set 1 Due Wednesday, February 1, 2012.

1. *Lorentz tensors*

Assume the matrix $\Lambda^{\mu}{}_{\nu}$ describes a Lorentz transformation, such that $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$.

a) If $T^{\mu\nu}$ and $B^{\mu\nu}$ are tensors under Lorentz transformations, prove that $T^{\mu\nu} B_{\nu\mu}$ and $T^{\mu\nu} B_{\mu\nu}$ are Lorentz scalars.

b) How does $T^{\mu\nu} B_{\mu\alpha}$ transform? What kind of tensor is this?

c) If $A_{\mu\nu}(x)$ is a tensor field, write down all Lorentz invariants that can be written as a product of two factors of either $A_{\mu\nu}(x)$ or its first derivatives.

d) Assume that the Minkowski metric, $\eta_{\mu\nu}$, transforms as a (0,2) tensor under Lorentz transformations. Show that $\eta_{\mu\nu}$ is invariant under Lorentz transformations.

2. *The Levi-Civita tensor*

The Levi-Civita tensor $\epsilon^{\mu\nu\lambda\sigma}$ is antisymmetric under exchange of any two of its indices, with $\epsilon^{0123} = +1$. Show that $\epsilon^{\mu\nu\lambda\sigma}$ is invariant under Lorentz transformations with detΛ=+1.

Note that the determinant of a 4×4 matrix A with components $A_{\mu\nu}$, where $\mu, \nu \in \{0, 1, 2, 3\}$, can be written

$$\det A = \sum_{\mu\nu\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} A_{0\mu} A_{1\nu} A_{2\lambda} A_{3\sigma}.$$

3. *Lorentz transformation of the electromagnetic field*

Maxwell's equations can be written in a Lorentz-covariant form in terms of the antisymmetric field-strength tensor $F^{\mu\nu}$. The components of $F^{\mu\nu}$

are:

$$\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix},$$

where E_i and B_i are the components of the electric and magnetic field, respectively.

Suppose $\mathbf{B}$=0 in some reference frame. Consider a Lorentz boost by speed v in the z-direction. By considering the Lorentz transformation of $F^{\mu\nu}$ determine the components of the electric field $\mathbf{E}'$ and magnetic field $\mathbf{B}'$ in the boosted frame in terms of the electric field $\mathbf{E}$ in the original frame and v.

4. *Lorentz invariants of electromagnetism*

In terms of $\mathbf{E}$ and $\mathbf{B}$, calculate $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}$.