Conservation of $T_{\mu\nu}$?

cf Feynman 5.5, 6.1

We have swept under the rug an inconsistency in the linearized theory of gravitation. We have assumed that $T_{\mu\nu}$ is conserved, i.e. $\partial_\mu T^{\mu\nu}=0$, which would imply that the energy of the system described by $T_{\mu\nu}$ is time independent. However, gravitational radiation carries off energy (it can do work), so either the total energy of gravity + matter is not conserved, or $T_{\mu\nu}$ for the matter is not itself conserved. It will be useful for us to explore conservation of $T_{\mu\nu}$ to understand the issue.

We have used for the $T_{\mu\nu}$ of matter,

$$T^{\mu\nu} = m \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,\delta^3(\vec{x}-\vec{x}(t))\frac{d\tau}{dt}$$

$$= m\int dt\, \delta^4(x-x(t))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{d\tau}{dt}$$

$$= m\int d\tau\, \delta^4(x-x(\tau))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\partial_\mu T^{\mu\nu} = m\int d\tau\, \underbrace{\partial_\mu \delta^4(x-x(\tau))\frac{dx^\mu}{d\tau}}_{-\frac{d}{d\tau}\delta^4(x-x(\tau))}\frac{dx^\nu}{d\tau}$$

$$\overset{\text{by parts}}{=} m\int d\tau\, \delta^4(x-x(\tau))\frac{d^2x^\nu}{d\tau^2}$$

$$= m\int d\tau\, \delta^4(x-x(\tau))\left(-\Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)$$

by eq. of motion

$$= -\Gamma^\nu_{\alpha\beta}T^{\alpha\beta}$$

$$\Rightarrow \boxed{\partial_\mu T^{\mu\nu} = -\Gamma^\nu_{\alpha\beta}T^{\alpha\beta}}$$

In our linear theory, $g_{\mu\nu} = \eta_{\mu\nu} - 2\lambda h_{\mu\nu}$.

$g^{\mu\nu} \approx \eta^{\mu\nu} + 2\lambda h^{\mu\nu}$ ← indices raised w/ $\eta^{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = -\Gamma^\nu_{\alpha\beta} T^{\alpha\beta}$$

$$\approx -\frac{1}{2}\eta^{\nu\rho}(-2\lambda)\left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}\right) T^{\alpha\beta} + \mathcal{O}(\lambda^2)$$

We can find tensors $\chi^{\mu\nu}$ bilinear in $h_{\mu\nu}$ so that

$$\partial_\mu\left(T^{\mu\nu} + \chi^{\mu\nu}\right) = 0, \quad \text{i.e.}$$

$$\boxed{\partial_\mu \chi^{\mu\nu} = -\lambda\,\eta^{\nu\rho}\left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}\right) T^{\alpha\beta} + \mathcal{O}(\lambda^2)}$$

Suppose the action contains higher-order terms in $h_{\mu\nu}$,

$$S_{grav} \supset \int d^4x \Big[\underbrace{a\, h_{\alpha\beta}\,\partial_\gamma h^{\alpha\beta}\,\partial_\delta h^{\gamma\delta} + \dots}_{L^{(3)}}\Big]$$

There are 18 independent third-order terms.

From the coupling $S \supset \lambda\int d^4x\, h_{\mu\nu} T^{\mu\nu}$,

$$T^{\mu\nu} = -\frac{1}{\lambda}\frac{\partial L}{\partial h_{\mu\nu}},$$

so we ask that $\chi^{\mu\nu} = -\frac{1}{\lambda}\frac{\partial L^{(3)}}{\partial h_{\mu\nu}}$.

The boxed equation for $\partial_\mu \chi^{\mu\nu}$ above then determines the 18 coefficients:

Feynman (6.1.13)

$$L^{(3)} = +\lambda\Big[h^{\alpha\beta}\bar{h}^{\gamma\delta}\partial_\gamma\partial_\delta\bar{h}_{\alpha\beta} + h_\gamma{}^{\beta}h^{\gamma\alpha}\partial_\delta\partial^\delta\bar{h}_{\alpha\beta}$$
$$-2h^{\alpha\beta}h_{\beta\delta}\partial^\gamma\partial^\delta\bar{h}_{\alpha\gamma} + 2\bar{h}_{\alpha\beta}\partial_\sigma\bar{h}^{\sigma\alpha}\partial_\rho\bar{h}^{\rho\beta}$$
$$+\left(\tfrac{1}{2}h_{\alpha\beta}h^{\alpha\beta} + \tfrac{1}{4}h_\alpha{}^{\alpha}h_\beta{}^{\beta}\right)\partial_\sigma\partial_\rho\bar{h}^{\sigma\rho}\Big]$$

This takes care of conservation of $T^{\mu\nu}$ to $\mathcal{O}(\lambda)$, as $\boxed{T^{\mu\nu}_{tot} \equiv -\frac{1}{\lambda}\frac{\partial L}{\partial h_{\mu\nu}}}$ satisfies

$$\partial_\mu T^{\mu\nu}_{tot} = \mathcal{O}(\lambda^2).$$

We can cancel the $\mathcal{O}(\lambda^2)$ terms by adding terms to L of $\mathcal{O}(h_{\mu\nu})^4$, etc.

The guiding principle is the eqn $\partial_\mu T^{\mu\nu}_{(grav)} = -\Gamma^\nu_{\alpha\beta}T^{\alpha\beta}_{(grav)}$

The solution to this, with the additional assumptions that $T^{\mu\nu}_{(grav)}$ is derived from an action as above and that there are only two derivatives of $h_{\mu\nu}$ in the action, is equivalent to Einstein's theory.

Einstein's approach is at this stage much more elegant, so we turn now to the geometrical approach to general relativity.

In the geometric approach we begin by assuming that all particles which are free except for the influence of gravitation fall along trajectories that stationarize the proper time, $d\tau^2 = -dx^\mu dx^\nu g_{\mu\nu}$.

These trajectories are <u>geodesics</u> in the spacetime described by the metric $g_{\mu\nu}$, and satisfy the <u>geodesic equation</u>

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0,$$ where τ parametrizes the trajectory.

As a result of gravity, $g_{\mu\nu}$ is not equivalent to a coordinate transformation of $\eta_{\mu\nu}$ in general, although at any point P we can choose $g_{\mu\nu} = \eta_{\mu\nu}$, and $\Gamma^\mu_{\alpha\beta} = 0$. These are the locally inertial, or freely falling coordinates at pt. P.

But how can we test whether $g_{\mu\nu}$ is just describing a coordinate transformation of $\eta_{\mu\nu}$ <u>everywhere</u>, i.e. whether or not there is gravitation?

Aside from the minus sign in $\eta_{00} = -1$, $\eta_{\mu\nu}$ is the Euclidean metric, so we think of $\eta_{\mu\nu}$ as describing a flat spacetime. Gravitation is described by the <u>curvature</u> of the spacetime.

To get a sense for geometric notions like curvature of spacetime, we will first describe the analogous geometric concepts in ordinary space.

Consider a 2 sphere

$$x^2+y^2+z^2=R^2 \rightarrow z=\sqrt{R^2-x^2-y^2}$$

$ds^2=dx^2+dy^2+dz^2$ infinitessimal length

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$=-\frac{(x\,dx+y\,dy)}{\sqrt{R^2-x^2-y^2}}$$

$$(dz)^2=\frac{x^2dx^2+y^2dy^2+2xy\,dx\,dy}{R^2-x^2-y^2}$$

$$ds^2=\left(1+\frac{x^2}{R^2-x^2-y^2}\right)dx^2+\left(1+\frac{y^2}{R^2-x^2-y^2}\right)dy^2+\left(\frac{2xy}{R^2-x^2-y^2}\right)dx\,dy$$

$$=g_{ij}dx^i dx^j$$

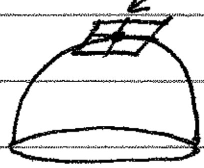
$$\boxed{g_{xx}=1+\frac{x^2}{R^2-x^2-y^2},\quad g_{yy}=1+\frac{y^2}{R^2-x^2-y^2} \qquad g_{xy}=\frac{xy}{R^2-x^2-y^2}}$$

valid for $x^2+y^2<R^2$, i.e. $z>0$ or $z<0$

At $z=1$, $x=y=0$,

$$\left.\begin{aligned} &g_{xx} = g_{yy} = 1, \quad g_{xy} = 0 \\ &\left.\frac{\partial g_{ij}}{\partial x^k}\right|_{x,y=0} = 0 \end{aligned}\right\} \text{locally flat coordinates.}$$

Cartesian Coordinates x, y



<u>Other coordinates:</u> $ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

$$g_{\theta\theta} = R^2, \quad g^{\theta\theta} = R^{-2}$$

$$g_{\varphi\varphi} = R^2\sin^2\theta, \quad g^{\varphi\varphi} = \frac{1}{R^2\sin^2\theta}$$

(Because g_{ij} is diagonal in these coordinates the inverse metric is easily determined.)

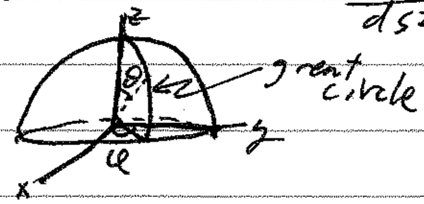
The nonvanishing Christoffel symbols are

$$\Gamma^{\theta}_{\varphi\varphi} = -\frac{1}{2}\sin 2\theta \qquad \text{(Exercise)}$$
$$\Gamma^{\varphi}_{\varphi\theta} = \cot\theta$$

$$\Rightarrow \left.\begin{aligned} &\frac{d^2\varphi}{ds^2} + (\cot\theta)\frac{d\varphi}{ds}\frac{d\theta}{ds} = 0 \\ &\frac{d^2\theta}{ds^2} + \left(-\frac{1}{2}\sin 2\theta\right)\left(\frac{d\varphi}{ds}\right)^2 = 0 \end{aligned}\right\} \text{Geodesic equations}$$

The solutions are the great circles, for example the great circles through the north pole,

$$\begin{cases} \varphi = \text{const} & \rightarrow \frac{d\varphi}{ds} = 0 = \frac{d^2\varphi}{ds^2} \\ \theta = \alpha s & \rightarrow \frac{d^2\theta}{ds^2} = 0 \end{cases}$$



If we lived on this 2-sphere, what evidence would we have of the non-flatness of space?

1) The distance between nearby geodesics would not be constant.
2) If we drew a circle (locus of pts equidistant from some pt) and measured the radius R, circumference C would not satisfy $C = 2\pi R$.

3) If we drew a triangle formed by 3 geodesics, the sum of the internal angles would not sum to 180°. In fact, the sum of the angles would grow with the area in the triangle.

We will make these notions of curvature precise. It will be helpful to describe geodesics in terms of parallely-transported vectors, and for that we will use tensor analysis. Now we will define tensors based on their transformation properties under general coordinate transformations.

weinberg 4.2

Suppose we change coordinates $x^{\mu} \to x'^{\mu}$.

$$V'^{\mu} = V^{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \quad \text{contravariant vector}$$

$$\left(\text{e.g. } dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \quad \text{chain rule}\right)$$

$$U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} U_{\nu} \quad \text{covariant vector}$$

$$\left(\text{e.g. } \frac{\partial \phi}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \phi}{\partial x^{\nu}}\right)$$

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \quad \text{coordinate transformation of metric.}$$

Mixed tensors: $T'^{\mu}{}_{\nu}{}^{\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} T^{\alpha}{}_{\beta}{}^{\gamma}$

Inverse metric: $g^{\lambda\mu}$, $g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu}$

$$g'^{\lambda\mu} \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\right) = \delta^{\lambda}_{\nu}$$

If $g'^{\lambda\mu} = \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} \frac{\partial x'^{\mu}}{\partial x^{\delta}} g^{\gamma\delta}$ then

$$g'^{\lambda\mu} g'_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\delta}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}}}_{\delta^{\alpha}_{\delta}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} g^{\gamma\delta}$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu} \quad \checkmark$$

$\Rightarrow$ $g^{\lambda\mu}$ is a contravariant tensor.

Tensors can be added, multiplied, and indices raised and lowered with the metric $g_{\mu\nu}$.

$$T^{\mu\nu} = g^{\nu\alpha} T^{\mu}{}_{\alpha} = g^{\nu\alpha} g^{\mu\beta} T_{\beta\alpha}$$

$$T_{\mu\nu} = g_{\mu\beta}\, g_{\nu\alpha}\, T^{\beta\alpha}$$

Indices can also be contracted, yielding tensors of lower rank:

$T_{\alpha\beta} \equiv T_{\alpha\beta}{}^{\mu}{}_{\mu}$ transforms like a rank-2 covariant tensor.

(Exercise)