cf. Weinberg 10.4

Generation of Gravitational Radiation

In the harmonic gauge the equations of motion were

$$\partial_\sigma \partial^\sigma h_{\alpha\beta} = \lambda \overline{T}_{\alpha\beta} = \lambda \left(T_{\alpha\beta} - \tfrac{1}{2} \eta_{\alpha\beta} T \right)$$

Weinberg calls this $S_{\alpha\beta}$.

Each component of $h_{\alpha\beta}$ satisfies an equation of the form

$\partial_\sigma \partial^\sigma \phi(x) = \rho(x)$, which can be solved using Green functions.

Suppose $G(x-x')$ satisfies $\boxed{\partial_\sigma \partial^\sigma G(x-x') = \delta^4(x-x')}$

Then $\boxed{\phi(x) = \int G(x-x')\, \rho(x')\, d^4x'}$ satisfies the Green function eqn. To solve the Green function eqn. we Fourier transform,

$$G(x-x') = \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k)\, e^{ik\cdot(x-x')}$$
$$\delta^4(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')}$$

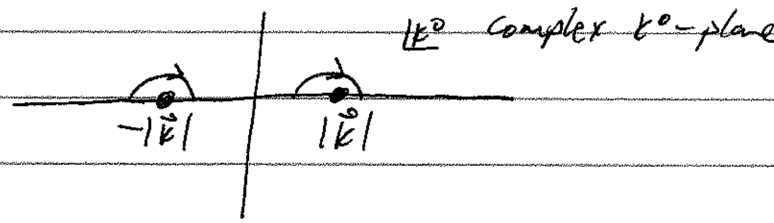
$$\Rightarrow \left(-k^\sigma k_\sigma\right) \tilde{G}(k) = 1$$

$$\boxed{\tilde{G}(k) = -\frac{1}{k^\sigma k_\sigma}} = -\frac{1}{k^2}$$

Then, $G(x-x') = -\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik\cdot(x-x')}$

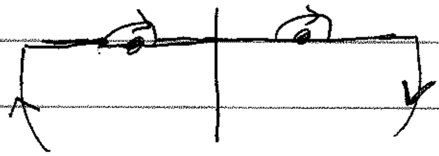
$$G(x-x') = -\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \, \frac{e^{-i[k^0(t-t') - \vec{k}\cdot(\vec{x}-\vec{x}')]}}{-(k^0)^2 + \vec{k}^2}$$

The k^0 integration has poles on the real axis, so we need to decide how to integrate around them.



Integrating above the poles gives the retarded Green function, which vanishes if $t<t'$. To see this, assume $t<t'$. The integrand is proportional to $e^{-ik^0(t-t')}$, so with $t<t'$ we can close the contour with a semicircle in the upper half plane. But the integration contour then does not enclose any poles, so it vanishes by Cauchy's theorem.

If $t>t'$ we close the contour in the lower half plane



The residue theorem gives

$$G(x-x') = -\int \frac{d^3k}{(2\pi)^3} \underbrace{(-i)}_{\text{clockwise}} \left\{ \frac{e^{-i[|\vec{k}|(t-t') - \vec{k}\cdot(\vec{x}-\vec{x}')]}}{-2|\vec{k}|} + \frac{e^{-i[-|\vec{k}|(t-t') - \vec{k}\cdot(\vec{x}-\vec{x}')]}}{2|\vec{k}|} \right\} \theta(t-t')$$

where $\theta(t-t') = 1$ if $t>t'$, 0 if $t<t'$.

$$G(x-x') = -\frac{i}{2}\int_0^\infty \frac{dk}{(2\pi)^3}k^2\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\left\{\frac{e^{-ik[t-t'-|\vec{x}-\vec{x}'|\cos\theta]}}{k}\right.$$

$$\left.-\frac{e^{ik[t-t'+|\vec{x}-\vec{x}'|\cos\theta]}}{k}\right\}\theta(t-t')$$

$$= \frac{-i\overbrace{(2\pi)}^{\varphi\text{ integral}}}{2(2\pi)^3}\int_0^\infty \frac{dk\,k^2}{k}\left\{\left(\frac{e^{-ik[t-t'-|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}-\frac{e^{-ik[t-t'+|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}\right)\right.$$

$$\left.+\left(-\frac{e^{ik[t-t'+|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}+\frac{e^{ik[t-t'-|\vec{x}-\vec{x}'|]}}{ik|\vec{x}-\vec{x}'|}\right)\right\}$$

$$\times\theta(t-t')$$

Taking $k\to -k$ in the second pair of integrals gives

$$G(x-x') = -\frac{1}{8\pi^2}\int_{-\infty}^{\infty} dk\left\{\frac{e^{-ik[t-t'-|\vec{x}-\vec{x}'|]}-e^{-ik[t-t'+|\vec{x}-\vec{x}'|]}}{|\vec{x}-\vec{x}'|}\right\}$$

$$\times\theta(t-t')$$

$$= -\frac{1}{8\pi^2|\vec{x}-\vec{x}'|}\cdot\left\{2\pi\,\delta\left(t-t'-|\vec{x}-\vec{x}'|\right)-2\pi\,\delta\left(t-t'+|\vec{x}-\vec{x}'|\right)\right\}$$

$$\times\theta(t-t')$$

$$\boxed{G(x-x') = -\frac{1}{4\pi|\vec{x}-\vec{x}'|}\,\delta\left(t-t'-|\vec{x}-\vec{x}'|\right)}$$

where we dropped the second term because it vanishes if $t>t'$, and the $\theta(t-t')$ is redundant because the delta-function enforces $t=t'+|\vec{x}-\vec{x}'|>t'$ in the first term.

We now have the (retarded) solution to the wave equation for $h_{\alpha\beta}$ in the presence of the source $\bar{T}_{\alpha\beta}$:

$$h_{\alpha\beta}(x) = -\frac{\lambda}{4\pi}\int d^4x' \, \frac{\bar{T}_{\alpha\beta}(x')}{|\vec{x}-\vec{x}'|}\,\delta(t-t'-|\vec{x}-\vec{x}'|)$$

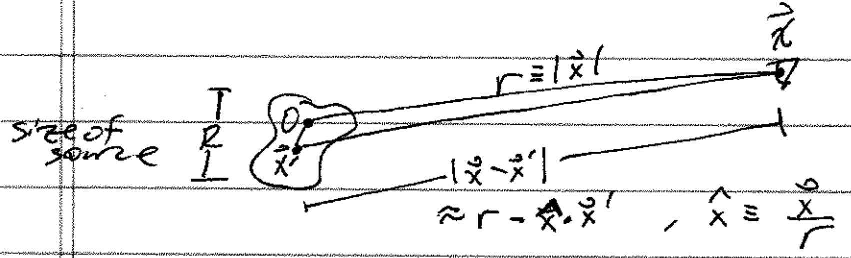
$$\boxed{h_{\alpha\beta}(x) = -\frac{\lambda}{4\pi}\int d^3x' \, \frac{\bar{T}_{\alpha\beta}(\vec{x}', t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}}$$

Suppose $T_{\mu\nu}(\vec{x},t)$ describes an oscillating source,

$$T_{\mu\nu}(\vec{x},t) = T_{\mu\nu}(\vec{x},\omega)\,e^{-i\omega t} + c.c.$$

Then $$h_{\alpha\beta}(x) = -\frac{\lambda}{4\pi}\int d^3x' \, \bar{T}_{\alpha\beta}(\vec{x}',\omega)\exp(-i\omega t + i\omega|\vec{x}-\vec{x}'|) + c.c.$$

The radiation zone is the region far from the source



$r \gg R$

$$h_{\alpha\beta}(x) \approx -\frac{\lambda}{4\pi}\frac{1}{r}\exp(i\omega r - i\omega t)\int d^3x' \, \bar{T}_{\alpha\beta}(\vec{x}',\omega)\exp(-i\omega\hat{x}\cdot\vec{x}') + c.c.$$

This is like a plane wave with $\vec{k} = \omega\hat{x}$, $k^0 = \omega$.

We can write

$$h_{\alpha\beta}(x) \approx \underbrace{\left(-\frac{\lambda}{4\pi r}\int d^3x' \; \bar{T}_{\alpha\beta}(\vec{x}',\omega)\, e^{-i\vec{k}\cdot\vec{x}'}\right)}_{\epsilon_{\alpha\beta}(\vec{x},\omega)} e^{ik\cdot x} + c.c.$$

$\vec{k} = \omega\hat{x}$

In terms of $\tilde{T}_{\alpha\beta}(\vec{k},\omega) \equiv \int d^3x' \; T_{\alpha\beta}(\vec{x}',\omega)\, e^{-i\vec{k}\cdot\vec{x}'}$,

$$\boxed{h_{\alpha\beta} = \underbrace{\left(-\frac{\lambda}{4\pi r}\left(\tilde{T}_{\alpha\beta}(\vec{k},\omega) - \frac{1}{2}\eta_{\alpha\beta}\tilde{T}(\vec{k},\omega)\right)\right)}_{\epsilon_{\alpha\beta}(\vec{x},\omega)} e^{ik\cdot x} + c.c.}$$

c.f. Weinberg 10.5

Quadrupolar Radiation

Suppose $\omega R \ll 1$ (nonrelativistic source)

where R = size of the source.

Then $\tilde{T}_{ij}(\vec{k},\omega) \approx \int d^3x \; T_{ij}(\vec{x},\omega)$ indep. of $\vec{k} = \omega\hat{x}$.

Conservation of angular momentum gives

$$\partial_\mu T^{\mu\nu} = \partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$\nu = j$: $\partial_0 T^{0j} + \partial_i T^{ij} = 0$ $\quad\Vert\; \nu = 0$: $\partial_0 T^{00} + \partial_i T^{i0} = 0$

∂_j: $\partial_0 \partial_j T^{0j} + \partial_j \partial_i T^{ij} = 0$

$-\partial_0 \partial_0 T^{00} + \partial_j \partial_i T^{ij} = 0$

with $T^{\mu\nu}(\vec{x},t) = T^{\mu\nu}(\vec{x},\omega)\, e^{-i\omega t} + c.c.$,

$$\boxed{\partial_j \partial_i T^{ij} = -\omega^2 T^{00}}$$

Multiply by $x^k x^l$, integrate over $\vec{x}$:

$$\int d^3x \; x^k x^l \, \partial_i \partial_j T^{ij} = -\omega^2 \int d^3x \; T^{00}(\vec{x},\omega) \, x^k x^l$$

∂_j by parts
$$= -2 \int d^3x \; x^k \, \partial_i T^{il}$$

∂_i by parts
$$= +2 \int d^3x \; T^{kl} \approx 2 \, \tilde{T}^{kl}(\vec{k},\omega)$$

$$\Rightarrow \boxed{\tilde{T}^{kl}(\vec{k},\omega) \approx -\frac{\omega^2}{2} D^{kl}(\omega)}$$

where $D^{kl}(\omega)$ is 1/3 x the Quadrupole Moment of the energy density,

$$\boxed{D^{kl}(\omega) \equiv \int d^3x \; T^{00}(\vec{x},\omega) \, x^k x^l}$$

Finally, we get the solution for $\bar{h}_{ij}$ in the radiation zone for nonrelativistic sources:

$$\boxed{\bar{h}_{ij} = +\frac{\lambda}{4\pi r} \frac{\omega^2}{2} \left(D_{ij}(\omega) e^{ik\cdot x} \right)} + \text{c.c.}$$

(where $\bar{h}_{ij} = h_{ij} - \frac{1}{2}\delta_{ij} h_\mu^{\ \mu}$)

This will be useful for evaluating the power emitted in gravitational radiation, but before we can do that we will need to know how to interpret the energy in the gravitational field. We will return to this later in order to consider the power radiated by a binary pulsar.