Motion of a Gravitating Particle

We have already written equations describing the motion of freely falling particles in an arbitrary coordinate system:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

where $d\tau^2 = -g_{\mu\nu}\,dx^\mu\,dx^\nu$ is the (proper time)^2,

and $\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}\right)$

The metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu_{\nu\lambda}$ contain information about the gravitational field.

We now have a theory for gravitation in terms of the rank-2 tensor field $h_{\mu\nu}$, which couples to matter through a term $-\lambda h_{\mu\nu}T^{\mu\nu}$ in the Lagrangian density.

For a relativistic particle, the energy is $mc^2\gamma = m\frac{dt}{d\tau}c^2$.

$d\tau^2 = -dt^2 + \frac{1}{c^2}dx^2$ here.

with $c=1$, $E = m\gamma$.

$$= \int d^3x\,\underbrace{m\frac{dt}{d\tau}\,\delta^3(\vec{x} - \vec{x}(t))}_{T^{00}}$$

The momentum is $p^i = m\frac{dx^i}{dt}\gamma = m\frac{dx^i}{dt}\frac{dt}{d\tau} = m\frac{dx^i}{d\tau}$

$$= \int d^3x\,\underbrace{m\frac{dx^i}{d\tau}\,\delta^3(\vec{x} - \vec{x}(t))}_{T^{0i}}$$

The term in the action $-\lambda\int h_{\mu\nu}T^{\mu\nu}\,d^3x\,dt$ includes

$$-\lambda\int h_{0\nu}T^{0\nu}d^3x\,dt = -\lambda\int h_{0\nu}\,m\frac{dx^\nu}{d\tau}\frac{dt}{d\tau}\,d\tau$$

integrating over x.

$$-\lambda\int h_{0\nu} T^{0\nu} d^3x\,dt = -\lambda\int h_{0\nu}\, m \frac{dx^\nu}{d\tau}\frac{dx^0}{d\tau}\, d\tau$$

$(x^0 = t)$

By Lorentz invariance, the action should then contain the term $-\lambda\int h_{\mu\nu} T^{\mu\nu} d^3x\,dt = -\lambda\int h_{\mu\nu}\, m \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\, d\tau$,

where we identify $\boxed{T^{\mu\nu} = m\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,\delta^3(\vec{x}-\vec{x}(t))\frac{d\tau}{dt}}$

In the absence of gravitation the equation of motion $m\frac{d^2x^\mu}{d\tau^2} = 0$ follows from an action

$$S_0 = \frac{1}{2}m\int d\tau\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,\eta_{\mu\nu}$$

Adding to this the gravitational coupling above, we postulate the action describing particle motion in a (weak) gravitational field:

$$S = \frac{1}{2}m\int d\tau\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,\eta_{\mu\nu} - \lambda m\int d\tau\, h_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$= \frac{1}{2}m\int d\tau\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\eta_{\mu\nu} - 2\lambda h_{\mu\nu}\right)$$

$$\equiv \boxed{\frac{1}{2}m\int d\tau\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,g_{\mu\nu}}$$

where $g_{\mu\nu} \equiv \eta_{\mu\nu} - 2\lambda h_{\mu\nu}$, $g_{\mu\nu} = g_{\nu\mu}$

The action is a functional of $x^\mu(\tau)$ and $\frac{dx^\mu}{d\tau}$. Stationarizing the action gives the equation of motion,

$$m\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}m\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial g_{\alpha\nu}}{\partial x^\mu} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \left((\partial_\alpha g_{\mu\nu})\frac{dx^\alpha}{d\tau}\right)\frac{dx^\nu}{d\tau} - \frac{1}{2}(\partial_\mu g_{\alpha\nu})\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\Big(\underbrace{\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha}}_{\text{symmetrizing in } \alpha\leftrightarrow\nu} - \partial_\mu g_{\alpha\nu}\Big)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Multiply by $g^{\beta\mu}$:

$$\frac{d^2x^\beta}{d\tau^2} + \underbrace{\frac{1}{2}g^{\beta\mu}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right)}_{\Gamma^\beta_{\alpha\nu}}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\boxed{\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}$$

We have recovered the equation for a freely falling particle with affine connection $\Gamma^\beta_{\alpha\nu}$!

<u>Note</u>: In the present discussion the metric is

$$g_{\mu\nu} = \eta_{\mu\nu} - 2\lambda h_{\mu\nu}.$$

The factor of (-2λ) is from the normalization of terms in the particle action. Had we started with the free particle action in the absence of gravity also multiplied by (-2λ), then we would have defined $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ as before.

Motion of particles in a gravitational wave

Suppose a pulse of gravitational radiation passes a collection of free particles.

Superposition of plane waves: $\left(k^\mu = (k,0,0,k) \right)$

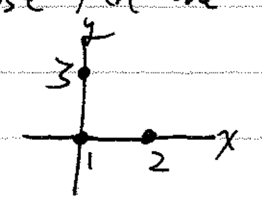
$$h_{\mu\nu} = \int dk\, \tilde{f}(k)\, e^{ik(z-t)}\, \epsilon_{\mu\nu} + \text{c.c.}$$

$$\equiv f(z-t)\, \epsilon_{\mu\nu} + \text{c.c.}$$

By a gauge choice we can set

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{xx} & \epsilon_{xy} & 0 \\ 0 & \epsilon_{xy} & -\epsilon_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Suppose there are three particles in the x-y plane:



$$\vec{x}_1 = (0,0,0)$$
$$\vec{x}_2 = (a,0,0)$$
$$\vec{x}_2 = (0,a,0)$$

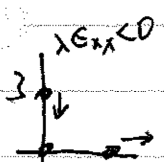
Proper time defines physical distance between points:

$$\Delta\tau_{12}^2 = g_{xx}\, a^2 = (1 - 2\lambda h_{xx})\, a^2$$

$$\Delta\tau_{12} = a\sqrt{1-2\lambda h_{xx}} \approx a\left(1 - \lambda h_{xx}\right) = a\left(1 - \lambda \epsilon_{xx} f(z-t)\right)$$

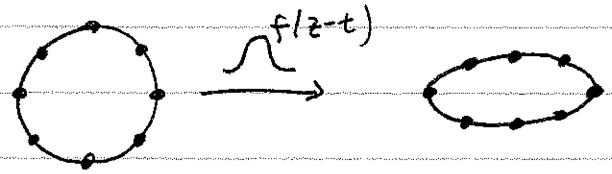
$$\Delta\tau_{13}^2 = g_{yy}\, a^2 = (1 - 2\lambda h_{yy})\, a^2$$

$$\Delta\tau_{13} = a\sqrt{1-2\lambda h_{yy}} \approx a\left(1 - \lambda h_{yy}\right) = a\left(1 + \lambda \epsilon_{xx} f(z-t)\right)$$

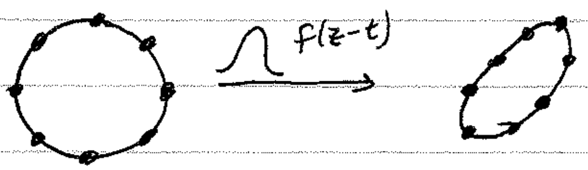


As the distance between 1 and 3 shrinks, the distance bet. 1 and 2 grows, and vice versa.

As the pulse of gravitational radiation passes, a circular distribution of particles is distorted into an ellipsoidal shape:



$\lambda \epsilon_{xx} < 0, \ \epsilon_{xy} = 0$



$\epsilon_{xx} = 0, \ \lambda \epsilon_{xy} < 0$

This distortion is the basis of gravitational wave searches.

(We will later have a less hand-wavy language to discuss the relative motion of nearby points, i.e. the <u>geodesic deviation</u>. We will wait until our discussion of spacetime curvature to return to this.)