It will be useful to define some notation. We will use a bar on a 2-index tensor to mean the following combination:

$$\boxed{\bar{A}_{\mu\nu} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{2}\eta_{\mu\nu} A^\sigma{}_\sigma}$$

For a symmetric tensor like $h_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\sigma{}_\sigma$$

Note: $\bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$

More notation: $\boxed{h \equiv h^\sigma{}_\sigma}$ "Trace", $\boxed{\bar{h}^\sigma{}_\sigma = -h}$

The equations of motion for $h_{\alpha\beta}$ take the form

$$\boxed{\begin{aligned} G_{\alpha\beta} \equiv & -\partial_\sigma \partial^\sigma h_{\alpha\beta} + (\partial_\beta \partial^\sigma h_{\alpha\sigma} + \partial_\alpha \partial^\sigma h_{\beta\sigma}) \\ & - (\partial_\alpha \partial_\beta h + \eta_{\alpha\beta} \partial_\mu \partial_\nu h^{\mu\nu}) + \eta_{\alpha\beta} \partial_\mu \partial^\mu h \\ & = -\lambda T_{\alpha\beta} \end{aligned}}$$

Taking the trace:

$$\begin{aligned} G \equiv & -\partial_\sigma \partial^\sigma h + 2\, \partial^\alpha \partial^\sigma h_{\alpha\sigma} - \partial_\alpha \partial^\alpha h + 4 \partial_\mu \partial^\mu h - 4 \partial_\mu \partial_\nu h^{\mu\nu} \\ & = 2\,(-\partial^\alpha \partial^\sigma h_{\alpha\sigma} + \partial_\alpha \partial^\alpha h) \\ & = -\lambda T \end{aligned}$$

$$
\begin{aligned}
\bar{G}_{\alpha\beta} &= G_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} G \\
&= -\partial_\sigma\partial^\sigma h_{\alpha\beta} + \partial_\beta\partial^\sigma h_{\alpha\sigma} + \partial_\alpha\partial^\sigma h_{\beta\sigma} - \partial_\alpha\partial_\beta h \\
= -\lambda \bar{T}_{\alpha\beta} &= -\lambda\left(T_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} T\right)
\end{aligned}
$$

These equations are invariant under the transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ for arbitrary functions $\xi_\mu(x)$:

$$
\begin{aligned}
\bar{G}_{\alpha\beta} \to \bar{G}_{\alpha\beta} &- \partial_\sigma\partial^\sigma\left(\cancel{\partial_\alpha \xi_\beta} + \cancel{\partial_\beta \xi_\alpha}\right) + \partial_\beta\partial^\sigma\left(\cancel{\partial_\alpha \xi_\sigma} + \cancel{\partial_\sigma \xi_\alpha}\right) \\
&+ \partial_\alpha\partial^\sigma\left(\cancel{\partial_\beta \xi_\sigma} + \cancel{\partial_\sigma \xi_\beta}\right) - \partial_\alpha\partial_\beta\left(\cancel{2\partial_\sigma \xi^\sigma}\right) \\
= \bar{G}_{\alpha\beta}
\end{aligned}
$$

Given a conserved $T_{\alpha\beta}$, for any solution to the equation for $h_{\alpha\beta}$ there are infinitely more solutions given by this transformation.

This invariance is reminiscent of gauge invariance in the relativistic formulation of electrodynamics. The equations of motion of electrodynamics can be written in terms of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
The field strength is invariant under $A_\mu \to A_\mu + \partial_\mu \theta(x)$ for arbitrary $\theta(x)$. Physical observables are independent of $\theta(x)$, so the gauge invariance reflects a redundancy in the description of the field $A_\mu(x)$.

Just as in electrodynamics, we can choose to fix the gauge by a condition on $h_{\mu\nu}$ that makes certain equations look simpler.

For example, we may choose to set

$$\boxed{\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h^\mu{}_\mu} \qquad \text{Harmonic gauge.}$$

To see that we can make this choice, assume

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h = f^\nu \neq 0.$$

Let $h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu$

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h \to \partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h + \partial_\mu \partial^\mu \xi^\nu + \partial_\mu \partial^\nu \xi^\mu - \frac{1}{2}\cdot 2\partial^\nu \partial_\mu$$

$$= 0 \quad \text{if} \quad \partial_\mu \partial^\mu \xi^\nu = -f^\nu,$$

which can be solved for ξ^ν.

Note that the harmonic gauge condition does not completely fix the gauge, because we can always add a solution to $\partial_\mu \partial^\mu \xi^\nu = 0$, the wave equation.

In the harmonic gauge, the equations of motion simplify:

$$\boxed{-\partial_\sigma \partial^\sigma h_{\alpha\beta} = -\lambda T_{\alpha\beta}}$$

Plane wave solutions

cf Weinberg Ch. 10

Consider the homogeneous equations

$$(*)\quad \begin{cases} \partial_\alpha \partial^\alpha h_{\mu\nu} = 0 \\ \partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\mu{}_\mu \end{cases}$$

Plane wave solutions have the form

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu}\exp(ik\cdot x) + \epsilon^*_{\mu\nu}\exp(-ik\cdot x)$$

where $k\cdot x \equiv k_\mu x^\mu$, $\epsilon_{\mu\nu} \equiv$ polarization tensor (in general complex)

The equations $(*)$ imply

$$\begin{cases} -k_\alpha k^\alpha h_{\mu\nu} = 0 \\ k_\mu \epsilon^\mu{}_\nu = \frac{1}{2}k_\nu \epsilon^\mu{}_\mu \end{cases}$$

Symmetry of $h_{\mu\nu}$ implies $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$.

Consider the gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ with $\xi^\mu(x) = i\mathcal{E}^\mu \exp(ik\cdot x) + i\mathcal{E}^{\mu *}\exp(-ik\cdot x)$

This is equivalent to transforming the polarization

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + k_\mu \mathcal{E}_\nu + k_\nu \mathcal{E}_\mu$$

This class of gauge transformations preserves the harmonic condition.

$$k_\mu \epsilon^\mu{}_\nu \to k_\mu \epsilon^\mu{}_\nu + \overset{0}{\vec{k}^2}\mathcal{E}_\nu + (k\cdot\mathcal{E})k_\nu$$

$$\frac{1}{2}k_\nu h^\mu{}_\mu \to \frac{1}{2}k_\nu h^\mu{}_\mu + (k\cdot\mathcal{E})k_\nu$$

Number of independent propagating solutions:

For each k^μ satisfying $k_\mu k^\mu = 0$,

$\epsilon_{\mu\nu}$ — symmetric 4×4 matrix $\rightarrow$ 10 components

harmonic condition -4

remaining gauge freedom $\underline{-4}$

2 independent polarizations.

Example: wave traveling in x^3-direction.

$$k^1 = k^2 = 0, \quad k^3 = k^0 = k > 0.$$

Harmonic condition:
$$\left.\begin{aligned} k^3\epsilon_{31} + k^0\epsilon_{01} &= k^3\epsilon_{32} + k^0\epsilon_{02} = 0 \\ k^3\epsilon_{33} + k^0\epsilon_{03} &= -(k^3\epsilon_{30} + k^0\epsilon_{00}) \\ &= \tfrac{1}{2}k^3(\epsilon_{11} + \epsilon_{22} + \epsilon_{33} - \epsilon_{00}) \end{aligned}\right\}$$

$k^3 = k^0 = k$
$$\Rightarrow \left.\begin{aligned} &\epsilon_{31} + \epsilon_{01} = \epsilon_{32} + \epsilon_{02} = 0 \\ &\epsilon_{33} + \epsilon_{03} = -(\epsilon_{30} + \epsilon_{00}) = \tfrac{1}{2}(\epsilon_{11} + \epsilon_{22} + \epsilon_{33} - \epsilon_{00}) \end{aligned}\right\}$$

$$\Rightarrow \boxed{\begin{aligned} &\epsilon_{01} = -\epsilon_{31}, \quad \epsilon_{02} = -\epsilon_{32}, \\ &\epsilon_{22} = -\epsilon_{11}, \quad \epsilon_{03} = -\tfrac{1}{2}(\epsilon_{33} + \epsilon_{00}) \end{aligned}}$$

$\epsilon_{01}, \epsilon_{02}, \epsilon_{22}, \epsilon_{03}$ dependent on other polarizations

Residual gauge freedom:
$$\left.\begin{aligned} \epsilon_{13} &\to \epsilon_{13} + k\xi_1 \\ \epsilon_{23} &\to \epsilon_{23} + k\xi_2 \\ \epsilon_{33} &\to \epsilon_{33} + 2k\xi_3 \\ \epsilon_{00} &\to \epsilon_{00} - 2k\xi_0 \end{aligned}\right\}$$
Do not have absolute physical significance.

$\Rightarrow$ Only two components $(\epsilon_{11}, \epsilon_{12})$ have absolute physical significance.

Helicity of Gravitational Waves

Consider a rotation by angle θ about the x^3-axis,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & +\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}, \quad (\Lambda^{-1})^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$$

The polarization transforms as $\epsilon_{\mu\nu} \to (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu \epsilon_{\alpha\beta} \equiv \epsilon'_{\mu\nu}$.

Defining $\epsilon_\pm \equiv \epsilon_{11} \mp i\epsilon_{12} = -\epsilon_{22} \mp i\epsilon_{12}$

$f_\pm \equiv \epsilon_{31} \mp i\epsilon_{32} = -\epsilon_{01} \pm i\epsilon_{02}$

It is straightforward to check that under the rotation,

$\epsilon'_\pm = \exp(\pm 2i\theta)\,\epsilon_\pm$ ← helicity ± 2

$f'_\pm = \exp(\pm i\theta)\, f_\pm$ ← helicity ± 1

$\epsilon'_{33} = \epsilon_{33}$, $\epsilon'_{00} = \epsilon_{00}$ ← helicity 0

Any plane wave which transforms as $\psi' = e^{ih\theta}\psi$ under a rotation by θ about the direction of motion is said to have helicity h.

Gravitational waves are decomposed into helicity 2, 1, 0 parts, but only the helicity 2 parts are physically significant as per our previous discussion.

Comparison w/ electromagnetic waves: In Lorenz gauge $\begin{cases} \partial_\mu \partial^\mu A_\alpha = 0 \\ \partial_\alpha A^\alpha = 0 \end{cases}$

Residual gauge freedom: $A^\mu \to A^\mu + \partial^\mu \theta$ s.t. $\partial_\mu \partial^\mu \theta = 0$

propagating degrees of freedom $= 4 - \underbrace{1}_{\text{Lorenz gauge condition}} - \underbrace{1}_{\text{residual gauge freedom}} = 2$

physical plane waves have helicity ± 1 in electromagnetism.