

## Scalar Gravity?

A first guess at a relativistic formulation of Newtonian gravity might be that gravitation is described by a scalar field.

Rather than  $\nabla^2 \phi = 4\pi G \rho$  as in Newtonian gravity, we might write the left-hand side in a Lorentz-covariant way as

$$-\frac{\partial^2}{\partial t^2} \phi + \nabla^2 \phi = 4\pi G \rho(\vec{x}, t) \quad (c=1)$$

This equation of motion, <sup>for  $\phi$</sup>  would follow from the Lagrangian density

$$L = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + 4\pi G \rho(\vec{x}, t) \phi$$

For static or slowly-varying sources  $\rho(\vec{x}, t)$ , the solution for  $\phi$  is determined by the Poisson equation of Newtonian gravity, but the field  $\phi$  propagates at a finite speed ( $c$ ) unlike in Newtonian gravity.

For static  $\rho(\vec{x})$  the Hamiltonian is that of Newtonian gravity if  $\rho(\vec{x})$  is identified with the mass density. The difficulty is that the mass density is not Lorentz invariant, so that the Lagrangian density is not a Lorentz scalar. It is difficult to accommodate the equivalence principle in this theory, as  $\rho(\vec{x}, t)$  should depend on the motion of the matter sourcing gravity, and gravitational binding energy as well.

Furthermore, the scalar theory of gravity has phenomenological problems. It predicts the wrong precession of planetary orbits and the wrong bending of light by the sun.

Hence, we will not pursue the scalar theory of gravity further.

### Vector Gravity?

We can entertain the possibility that gravity is mediated by a vector field, like electromagnetism. The immediate problem is that likes repel in electromagnetism, so we will not pursue a vector theory of gravity further.

### Tensor Gravity

The next-simplest possibility is that gravity is mediated by a two-index tensor field  $h_{\mu\nu}(\vec{x}, t)$ .

We can anticipate a coupling in the Lagrangian density proportional to  $h_{\mu\nu} T^{\mu\nu}$ , where  $T^{\mu\nu}$  is the energy-momentum tensor.  $T^{00}$  is the energy density, so the term  $h_{00} T^{00}$  replaces  $\phi\rho$ , and we anticipate that the other components of  $h_{\mu\nu}$  will be unimportant in the nonrelativistic weak-field limit.

We need to be more specific regarding the type of rank-2 tensor field we want to consider.

Lorentz transformations leave invariant the symmetric and antisymmetric components of tensors.

For example, if  $h_{\mu\nu} = h_{\nu\mu}$ ,

$$h'_{\alpha\beta} = (\Lambda^\mu{}_\alpha) (\Lambda^\nu{}_\beta) h_{\mu\nu} = (\Lambda^\mu{}_\alpha) (\Lambda^\nu{}_\beta) h_{\nu\mu} \\ = h'_{\beta\alpha}$$

Similarly, if  $h_{\mu\nu} = -h_{\nu\mu}$ ,

$$h'_{\alpha\beta} = (\Lambda^\mu{}_\alpha) (\Lambda^\nu{}_\beta) h_{\mu\nu} = (\Lambda^\mu{}_\alpha) (\Lambda^\nu{}_\beta) (-h_{\nu\mu}) \\ = -h'_{\beta\alpha}$$

The coupling  $h_{\mu\nu} T^{\mu\nu}$  would vanish for symmetric energy-momentum tensor  $T^{\mu\nu}$  and antisymmetric  $h_{\mu\nu}$ , so we guess that a symmetric  $h_{\mu\nu}$  will do the job.

### Lagrangian Density for a Symmetric Tensor Field

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ch. 4

For the equations of motion to have second-order spacetime derivatives we consider products of first derivatives in the Lagrangian. There are five independent possibilities:

$$\begin{aligned} & \partial_\sigma h_{\mu\nu} \partial^\sigma h^{\mu\nu} \\ & \partial_\sigma h_{\mu\nu} \partial^\nu h^{\mu\sigma} \\ & \partial_\nu h^{\mu\nu} \partial_\sigma h^{\sigma\mu} \\ & \partial_\nu h^{\mu\nu} \partial_\mu h^{\sigma\sigma} \\ & \partial^\mu h^\nu{}_\nu \partial_\mu h^\sigma{}_\sigma \end{aligned}$$

The second and third possibilities are related by integration by parts in the action, so they are redundant:

$$\begin{aligned}\int d^4x (\partial_\sigma h_{\mu\nu}) (\partial^\nu h^{\mu\sigma}) &= - \int d^4x h_{\mu\nu} \partial^\nu \partial_\sigma h^{\mu\sigma} \\ &= \int d^4x \partial^\nu h_{\mu\nu} \partial_\sigma h^{\mu\sigma} \\ &= \int d^4x \partial_\nu h^{\mu\nu} \partial_\sigma h^\sigma{}_\mu,\end{aligned}$$

where we dropped surface terms by assuming that  $h_{\mu\nu}$  falls off quickly enough at infinity.

The Lagrangian could also include terms without derivatives, like  $h_{\mu\nu} h^{\mu\nu}$  and  $h_{\mu}{}^\mu h^\nu{}_\nu$ .

These are mass terms, and would lead to a gravitational field that falls exponentially faster than the Coulomb potential, as the Yukawa potential, in the nonrelativistic limit. Hence we will not add such terms (though we may consider their effects later).

Terms linear in  $h_{\mu}{}^\mu$  are sources, which we do not add to the Lagrangian because we want  $h_{\mu\nu}$  to be allowed to vanish in the absence of physical sources of gravity.

So, we consider the action

$$S = \int d^4x \left[ a (\partial_\sigma h_{\mu\nu}) (\partial^\sigma h^{\mu\nu}) + b (\partial_\nu h^{\mu\nu}) (\partial_\sigma h^\sigma{}_\mu) \right. \\ \left. + c (\partial_\nu h^{\mu\nu}) (\partial_\mu h^\sigma{}_\nu) + d (\partial_\mu h^\nu{}_\nu) (\partial^\mu h^\sigma{}_\sigma) \right. \\ \left. - \lambda h_{\mu\nu} T^{\mu\nu} \right]$$

for some constants  $a, b, c, d, \lambda$ .

Varying  $S$  with respect to  $h_{\alpha\beta}$  gives the Euler-Lagrange eqs, and for each component of  $h_{\alpha\beta}$ . Keeping track of the symmetry of  $h_{\alpha\beta}$ , we can write the Euler-Lagrange equations as:

$$\frac{1}{2} \partial_\gamma \frac{\partial \mathcal{L}}{\partial (\partial_\gamma h_{\alpha\beta})} + \frac{1}{2} \partial_\gamma \frac{\partial \mathcal{L}}{\partial (\partial_\gamma h_{\beta\alpha})} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial h_{\alpha\beta}} + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial h_{\beta\alpha}}$$

$$\text{Use } \frac{\partial h_{\mu\nu}}{\partial h_{\alpha\beta}} = \delta_{\mu\alpha} \delta_{\nu\beta}, \quad \frac{\partial (\partial_\sigma h_{\mu\nu})}{\partial (\partial_\gamma h_{\alpha\beta})} = \delta_\sigma^\gamma \delta_{\mu\alpha} \delta_{\nu\beta}$$

(we can add  $\alpha \leftrightarrow \beta$  to each of these to take into account the symmetry of  $h_{\alpha\beta}$ , but we are already taking into account the symmetry of  $h_{\alpha\beta}$  in the way we write the Euler-Lagrange equations, so adding  $\alpha \leftrightarrow \beta$  a second time is redundant.)

For example,

$$\frac{\partial}{\partial (\partial_\gamma h_{\alpha\beta})} \left( a (\partial_\sigma h_{\mu\nu}) (\partial^\sigma h^{\mu\nu}) \right) = \frac{\partial}{\partial (\partial_\gamma h_{\alpha\beta})} \left( a (\partial_\sigma h_{\mu\nu}) (\partial^\sigma h^{\mu\nu}) \right) \\ \times \gamma^{\sigma\lambda} \gamma^{\mu\rho} \gamma^{\nu\delta}$$

$$= a \gamma^{\sigma\lambda} \gamma^{\mu\rho} \gamma^{\nu\delta} \left( \delta_\sigma^\gamma \delta_{\mu\alpha} \delta_{\nu\beta} \partial_\lambda h^{\rho\delta} + \partial_\sigma h_{\mu\nu} \delta_\lambda^\gamma \delta_\rho^\alpha \delta_\delta^\beta \right) \\ = 2a \partial^\gamma h_{\alpha\beta}$$

The Euler-Lagrange equations becomes:

$$\begin{aligned}
 & a \cdot 2 \partial_\sigma \partial^\sigma h_{\alpha\beta} + b (\partial^\sigma \partial_\beta h_{\alpha\sigma} + \partial^\sigma \partial_\alpha h_{\beta\sigma}) \\
 & + c (\partial_\alpha \partial_\beta h^\sigma{}_\sigma + \gamma_{\alpha\beta} \partial_\mu \partial_\nu h^{\mu\nu}) + d \cdot 2 \gamma_{\alpha\beta} \partial_\mu \partial^\mu h^\sigma{}_\sigma \\
 & = -\lambda T_{\alpha\beta}
 \end{aligned}$$

$T_{\alpha\beta}$  is conserved,  $\partial^\beta T_{\alpha\beta} = 0$ . Taking the derivative  $\partial^\beta$  on both sides of the Euler-Lagrange equations gives:

$$\begin{aligned}
 & 2a \partial_\sigma \partial^\sigma \partial^\beta h_{\alpha\beta} + b (\partial^\sigma \partial_\beta \partial^\beta h_{\alpha\sigma} + \partial^\sigma \partial_\alpha \partial^\beta h_{\beta\sigma}) \\
 & + c (\partial_\alpha \partial_\beta \partial^\beta h^\sigma{}_\sigma + \gamma_{\alpha\beta} \partial^\beta \partial_\mu \partial_\nu h^{\mu\nu}) + 2d \gamma_{\alpha\beta} \partial_\mu \partial^\mu \partial^\beta h^\sigma{}_\sigma = 0 \\
 & = \partial_\sigma \partial^\sigma \partial^\beta h_{\alpha\beta} (2a + b) + \partial^\sigma \partial^\beta \partial_\alpha h_{\beta\sigma} (b + c) \\
 & + \partial_\alpha \partial_\beta \partial^\beta h^\sigma{}_\sigma (c + 2d) = 0
 \end{aligned}$$

We can make the left-hand side identically vanish if  $b = -2a = -c = 2d$ , for example

$$a = -\frac{1}{2}, \quad b = +1, \quad c = -1, \quad d = +\frac{1}{2}$$

The action becomes:

$$\begin{aligned}
 S = \int d^4x \left[ \right. & \frac{1}{2} (\partial_\sigma h_{\mu\nu}) (\partial^\sigma h^{\mu\nu}) + (\partial_\nu h^{\mu\nu}) (\partial_\sigma h^\sigma{}_\mu) \\
 & - (\partial_\nu h^{\mu\nu}) (\partial_\mu h^\sigma{}_\sigma) + \frac{1}{2} (\partial_\mu h^\nu{}_\nu) (\partial^\mu h^\sigma{}_\sigma) \\
 & \left. - \lambda h_{\mu\nu} T^{\mu\nu} \right]
 \end{aligned}$$