Comparison Between Stationarization of Proper Time and Lagrangian Formalism

We have seen that the proper time elapsed along a trajectory is stationarized for trajectories that satisfy the equation of motion (in the absence of forces other than gravity).

By comparison with the weak-field Newtonian limit we learned that the (00)-component of the metric is related to the gravitational potential $\phi(\vec{x})$ in that limit via $g_{00} \approx -(1+2\phi)$.

We consider ϕ as small (weak field), and $\left|\frac{d\vec{x}}{dt}\right| \ll 1$ (slow compared to light).

The proper time in this limit is

$$T(A\to B) \approx \int_A^B \sqrt{dt^2(1+2\phi) - d\vec{x}^2}$$

← corrections are higher order in $\left(\frac{d\vec{x}}{dt}\right)\phi$

$$= \int_A^B \sqrt{(1+2\phi) - \left(\frac{d\vec{x}}{dt}\right)^2}\; dt$$

$$\approx \int_A^B \left(1 + \phi - \frac{1}{2}\left(\frac{d\vec{x}}{dt}\right)^2\right) dt$$ (expanding the square root about 1)

Multiplying by $(-m)$, where m is the particle's mass,

$$-mT(A\to B) \approx \int_A^B \left(\frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2 - m\phi - m\right) dt$$

If $T(A \to B)$ is stationarized along the trajectory, so is $-m\,T(A \to B)$.

We recognize $-mT(A \to B)$ as the nonrelativistic action for a particle moving in a gravitational field specified by $\phi(\vec{x})$, up to the addition of the irrelevant constant $-m\int_A^B dt$.

$$S \equiv -mT(A \to B) \approx \int_A^B L\,dt + \underbrace{\text{const.}}_{-m(t_B - t_A)}$$

where $\boxed{L = \frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2 - m\phi}$

For example, for a particle in a uniform gravitational field with gravitational acceleration $\vec{g} = g\hat{z}$,

$$\phi = gz, \text{ and}$$
$$L = \frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2 - mgz$$

In this case, $T(A \to B) \approx \int_A^B \left[(1+2gz) - \left(\frac{d\vec{x}}{dt}\right)^2\right]^{1/2} dt$

The proper time is larger if the trajectory spends time at larger z, but in order to reach larger z, $\left(\frac{dz}{dt}\right)^2$ must also be larger somewhere along the trajectory, which reduces T. The parabolic trajectory that <u>maximizes</u> T is a compromise between minimizing $\left(\frac{d\vec{x}}{dt}\right)^2$ while maximizing z.

The Principle of General Covariance

If we knew what the locally inertial frames were, we could write the equations of motion for a freely falling particle in an arbitrary coordinate system. Alternatively, if we knew the metric $g_{\mu\nu}$ in a coordinate system, we could write the equations of motion using the relation between $\Gamma^{\mu}_{\nu\lambda}$ and $g_{\mu\nu}$.

To understand gravitation, we want to find equations that determine $g_{\mu\nu}$ in terms of the sources of gravity, which in the Newtonian limit is the mass density.

The equations that determine $g_{\mu\nu}$ should be consistent with the principle of equivalence, which for our present purposes is more conveniently reformulated as the <u>principle of general covariance</u>:

(cf. Weinberg 4.1)

A physical equation holds in a general gravitational field if two conditions are met:

1. The eqn holds in the absence of gravity, i.e. if $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\lambda} = 0$.
2. The eqn is <u>generally covariant</u>, i.e. it preserves its form under a general coordinate transformation $x \to x'$.

According to the principle of equivalence, at any point there is a locally inertial coordinate system in which the effects of gravitation are absent. Condition (1) implies that the equation in question is valid in that coordinate system.
Condition (2) then implies that the equation is valid in any other coordinate system.

Note: General covariance is not a symmetry principle like Lorentz invariance. It is a statement about the effects of gravitation, but nothing else.

Note: The principle of equivalence only guarantees that an inertial coordinate system in the neighborhood of a point can be constructed such that the effects of gravitation are absent. Hence, condition (1) does not apply over distances large compared to the typical distances in the gravitational field, so we cannot use general coordinate invariance to determine the notion of a "large" object directly. Instead we should consider each infinitesimal element of a large object independently.

Coordinate Transformation of the metric

$d\tau^2 = -dx^\mu g_{\mu\nu} dx^\nu$

Suppose we consider a new coordinate system x'^μ,

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} dx'^\alpha$$

$$d\tau^2 = -\left(dx'^\alpha \frac{\partial x^\mu}{\partial x'^\alpha}\right) g_{\mu\nu} \left(\frac{\partial x^\nu}{\partial x'^\beta} dx'^\beta\right)$$

$$= -dx'^\alpha g'_{\alpha\beta} dx'^\beta$$

where $\boxed{g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}}$

The inverse metric $g^{\mu\nu}$ satisfies $g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda{}_\nu$.

Consider $\left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x'^\mu}{\partial x^\sigma} g^{\rho\sigma}\right) g'_{\mu\nu}$

$$= \left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x'^\mu}{\partial x^\sigma} g^{\rho\sigma}\right)\left(\frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial x^\eta}{\partial x'^\nu} g_{\kappa\eta}\right)$$

$$= \frac{\partial x'^\lambda}{\partial x^\rho} g^{\rho\kappa} \frac{\partial x^\eta}{\partial x'^\nu} g_{\kappa\eta} \quad \text{using } \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\kappa}{\partial x'^\mu} = \delta^\kappa{}_\sigma$$

$$= \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu} \quad \text{using } g^{\rho\kappa} g_{\kappa\eta} = \delta^\rho{}_\eta$$

$= \delta^\lambda{}_\nu$. Hence, we identify $\boxed{g'^{\lambda\mu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x'^\mu}{\partial x^\sigma} g^{\rho\sigma}}$

In a weak gravitational field we can write
$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1.$

Consider the infinitesimal coordinate transformation
$$x^\mu = x'^\mu + \epsilon^\mu(x'), \qquad |\epsilon^\mu| \ll 1$$

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x(x'))$$

$$= \left(\delta^\alpha_\mu + \frac{\partial \epsilon^\alpha}{\partial x'^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial \epsilon^\beta}{\partial x'^\nu}\right)\left(g_{\alpha\beta}(x') + \frac{\partial g_{\alpha\beta}}{\partial x'^\lambda}\epsilon^\lambda + \mathcal{O}(\epsilon^2)\right)$$

$$= g_{\mu\nu} + g_{\alpha\beta}\left(\frac{\partial \epsilon^\alpha}{\partial x'^\mu}\delta^\beta_\nu + \frac{\partial \epsilon^\beta}{\partial x'^\nu}\delta^\alpha_\mu\right) + \epsilon^\lambda \delta^\alpha_\mu \frac{\partial g_{\alpha\beta}}{\partial x'^\lambda}\delta^\beta_\nu + \mathcal{O}(\epsilon^2)$$

$$\boxed{g'_{\mu\nu} = g_{\mu\nu} + g_{\alpha\nu}\frac{\partial \epsilon^\alpha}{\partial x'^\mu} + g_{\beta\mu}\frac{\partial \epsilon^\beta}{\partial x'^\nu} + \frac{\partial g_{\mu\nu}}{\partial x'^\lambda}\epsilon^\lambda + \mathcal{O}(\epsilon^2)}$$

$$= \eta_{\mu\nu} + h_{\mu\nu} + \eta_{\alpha\nu}\frac{\partial \epsilon^\alpha}{\partial x'^\mu} + \eta_{\beta\mu}\frac{\partial \epsilon^\beta}{\partial x'^\nu} + \mathcal{O}(\epsilon h, \epsilon^2)$$

$$= \eta_{\mu\nu} + \underbrace{h_{\mu\nu} + \frac{\partial \epsilon_\nu}{\partial x'^\mu} + \frac{\partial \epsilon_\mu}{\partial x'^\nu}} + \mathcal{O}(\epsilon h, \epsilon^2)$$

$$\boxed{h'_{\mu\nu} = h_{\mu\nu} + \frac{\partial \epsilon_\nu}{\partial x'^\mu} + \frac{\partial \epsilon_\mu}{\partial x'^\nu}}$$

where $\epsilon_\mu = \eta_{\beta\mu}\epsilon^\beta$.

General coordinate invariance suggests that for weak fields, the equations determining $h_{\mu\nu}$ should be invariant under the infinitesimal transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$.