Review of Lagrangian Mechanics

Suppose a system is described by a set of N coordinates $q_i(t)$, $i=1,\dots,N$.

Action Principle: There exists a functional of q_i, $\dot{q}_i$ and t, called the action, which is stationary about variations $\delta q_i(t)$ along a classical path.

Lagrangian $L(q_i, \dot{q}_i, t)$

Action $S = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t)$

Variation $q_i(t) \to q_i(t) + \delta q_i(t)$

$\delta q_i(t_1) = \delta q_i(t_2) = 0$ (Boundaries fixed)

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right]$$

$$= \int_{t_1}^{t_2} dt \sum_i \delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right]$$

$\delta S = 0$ for arbitrary small variation $\delta q_i(t)$ along a classical path

$$\Longrightarrow \quad \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0} \quad \text{Euler-Lagrange Eqs.}$$

Hamiltonian Formulation

Given a Lagrangian L, define the *canonical momenta*

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \rightsquigarrow \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$ from Euler-Lagrange Eqs.

Define the *Hamiltonian* $H(p_i, q_i, t) \equiv \sum_i p_i \dot{q}_i - L$

H must be written in terms of p's and q's, *not* the $\dot{q}$'s, and the p's and q's must be independent. (This is not always possible!)

Vary the coordinates and momenta:

$$dH = \sum_i \left(dp_i\, \dot{q}_i + \cancel{p_i\, d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} \right)$$

$$= \sum_i \left(\dot{q}_i\, dp_i - \dot{p}_i\, dq_i \right)$$ using the Euler-Lagrange Eqs.

$$= \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$$

$$\rightsquigarrow \boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i \,, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}$$ Hamilton's Eqs.

If $\frac{\partial L}{\partial t} = 0 \rightsquigarrow \frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right) - \cancel{\frac{\partial L}{\partial t}}$

$$= \sum_i \left(\dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i \right) = 0$$

In that case H is called the *energy*, and is conserved.

Symmetries and Conservation Laws

Noether's Theorem: For every global symmetry parametrized by a continuous parameter there is a corresponding conservation law.

Under a variation of $q_i(t)$,

$$L \to L + \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$= L + \sum_i \left(\dot{p}_i \, \delta q_i + p_i \, \delta \dot{q}_i \right) \quad \text{using the E-L Eqs.}$$

$$= L + \frac{d}{dt}\left(\sum_i p_i \, \delta q_i \right) \quad \text{using } \delta \dot{q}_i = \frac{d}{dt} \delta q_i$$

Now consider a class of variations parametrized by a continuous parameter λ: $\delta q_i(t,\lambda) = \lambda \left.\frac{\partial q_i}{\partial \lambda}\right|_{\lambda=0}$ as $\lambda \to 0$

Suppose under this class of transformations

$L \to L + \lambda \frac{dF}{dt}$ for some $F(q_i, \dot{q}_i, t)$, $F(t_2) - F(t_1) = 0$
(not using the eqs. of motion)

Then the class of transformations $q_i(t) \to q_i(t,\lambda)$ is called a symmetry.

Hamilton's principle: $0 = \delta S = \int_{t_1}^{t_2} dt \, \delta L = \int_{t_1}^{t_2} dt \, \lambda \frac{dF}{dt}$

$$= \lambda \left(F(t_2) - F(t_1) \right) = 0$$

Such a symmetry transformation does not change the action or the equations of motion.

Around the eqs. of motion $L \to L + \frac{d}{dt}\left(\sum_i p_i \, \delta q_i(t,\lambda)\right)$

$$= L + \lambda \frac{d}{dt}\left(\sum_i p_i \frac{\partial q_i}{\partial \lambda}\right)\Big|_{\lambda=0} \text{ for } \lambda \ll 1$$

$$= L + \lambda \frac{dF}{dt} \text{ by assumption.}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\sum_i p_i \frac{\partial q_i}{\partial \lambda}\Big|_{\lambda=0} - F(q_i, \dot{q}_i, t)\right) = 0}$$

The quantity $Q \equiv \sum_i p_i \frac{\partial q_i}{\partial \lambda}\Big|_{\lambda=0} - F(q_i, \dot{q}_i, t)$

is conserved.

Example: ~~Space~~ Space translation of point particles

Lagrangian $L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - \sum_{ij} V_{ij}(|\vec{r}_i - \vec{r}_j|)$

$\vec{r}_i \to \vec{r}_i + \vec{a}\lambda$ takes $L \to L$ for any fixed $\vec{a}$.

$\frac{\partial \vec{r}_i(\lambda)}{\partial \lambda} = \vec{a}$, $F = 0$, $\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{r}}_i} = m_i \dot{\vec{r}}_i$

$Q = \sum_i \vec{p}_i \cdot \vec{a}$ is conserved. Since this is true $\forall \vec{a}$,
$\sum_i \vec{p}_i$ is conserved.

Translation invariance → Momentum Conservation

Example: Time translations $q_i(t) \to q_i(t+\lambda)$

$$\frac{\partial q_i(t,\lambda)}{\partial \lambda}\Big|_{\lambda=0} = \frac{d q_i(t)}{dt}$$

$$\delta L = \lambda \frac{dL}{dt} \quad \rightsquigarrow \quad F = L$$

$$Q = \sum_i P_i \dot{q}_i - L = H$$ is conserved.

Time translation invariance → Energy conservation

Classical Field Theory

$q_i(t) \rightsquigarrow \phi_i(\vec{x}, t)$ infinite set of generalized coordinates

$t \rightsquigarrow t$

$i \rightsquigarrow i, \vec{x}$

$\sum_i \rightsquigarrow \sum_i \int d^3\vec{x}$

Lagrangian $L(q_i, \dot{q}_i, t) \rightsquigarrow$ Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i, x^\mu)$

$L \equiv \int d^3x\, \mathcal{L}$ integrated over a spacelike time slice.

We assume that $\mathcal{L}$ is local in space and time, and depends on at most first derivatives w.r.t. $t, \vec{x}$.

If $\mathcal{L}$ is a Lorentz scalar, then the Euler-Lagrange eqs will be Lorentz covariant.

$$\delta S = \sum_i \int \underbrace{dt\, d^3x}_{d^4x} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}}_{\equiv \Pi^\mu_i} \underbrace{\delta(\partial_\mu \phi_i)}_{= \partial_\mu \delta\phi_i} \right]$$

under arbitrary variations $\delta\phi_i$ such that $\delta\phi_i(t_1, \vec{x}) = \delta\phi_i(t_2, \vec{x}) = 0$.

Integrate by parts $\rightsquigarrow \delta S = \sum_i \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \Pi^\mu_i \right] \delta\phi_i = 0.$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \Pi^\mu_i = \boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0}$$

Euler-Lagrange Eqs.

The canonical momentum $\overset{\text{of field } \phi_i}{\wedge}$ is $\Pi_i(t,\hat{x}) \equiv \Pi_i^0(t,\hat{x})$

(Don't think of Π^μ_i as a 4-vector generalization of the canonical momentum.)

$$\text{Hamiltonian } H = \int d^3x \underbrace{\left[\sum_i \Pi_i \, \partial_0 \phi_i - \mathcal{L}\right]}_{\text{Hamiltonian density } \mathcal{H} \equiv \left[\sum_i \Pi_i \partial_0 \phi_i - \mathcal{L}\right]}$$

Example: Most general $\mathcal{L}$ satisfying:

(1) $\mathcal{L}$ is a Lorentz scalar.

(2) $\mathcal{L}$ built from one real scalar field $\phi = \phi^*$

(3) $\mathcal{L}$ quadratic in ϕ, $\partial_\mu \phi$

↖ for linear eqs. of motion

$$\mathcal{L} = \frac{1}{2} a \left[\partial_\mu \phi \, \partial^\mu \phi + b \phi^2\right]$$

we can rescale $\phi \to \frac{\phi}{\sqrt{|a|}}$, $\mathcal{L} \to \pm\frac{1}{2}\left[\partial_\mu\phi\,\partial^\mu\phi + b\phi^2\right]$

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}\left(\pm\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi \pm \frac{1}{2} b\phi^2\right)$$

Use $\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)} = \delta^\mu_\alpha$

$$\begin{aligned}\Pi^\mu &= \pm\frac{1}{2}\eta^{\alpha\beta}\left(\delta^\mu_\alpha(\partial_\beta\phi) + (\partial_\alpha\phi)\delta^\mu_\beta\right) \\ &= \pm\frac{1}{2}\left(\eta^{\mu\beta}(\partial_\beta\phi) + \eta^{\alpha\mu}(\partial_\alpha\phi)\right) \\ &= \pm\partial^\mu\phi\end{aligned}$$

$$\partial_\mu \Pi^\mu - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow \boxed{\pm\left(\partial_\mu\partial^\mu\phi - b\phi\right) = 0}$$

$$\mathcal{H} = \pi^0 \partial_0 \phi - \mathcal{L}$$

$$= (\pm \partial^0 \phi)(\partial_0 \phi) \mp \frac{1}{2}\left[(\partial^0 \phi)(\partial_0 \phi) + (\partial^i \phi)(\partial_i \phi) + b \phi^2\right]$$

$$= \pm \frac{1}{2}(\partial^0 \phi)(\partial_0 \phi) \mp \frac{1}{2}(\partial^i \phi)(\partial_i \phi) \mp \frac{1}{2} b \phi^2$$

$$= \mp \frac{1}{2}(\partial_0 \phi)^2 \mp \frac{1}{2}(\nabla \phi)^2 \mp \frac{1}{2} b \phi^2$$

$\mathcal{H}$ is positive semi-definite if:

↑ can be zero

1) Choose the -ve sign in $\mathcal{L}$,

2) $b > 0$.

$$\longrightarrow \boxed{\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}$$

More Classical Field Theory

Example: Two real scalar fields $\phi_1(x)$, $\phi_2(x)$.

Most general Lagrangian density satisfying:

1) $\mathcal{L}$ is a Lorentz scalar
2) $\mathcal{L}$ is quadratic in $\phi_1, \phi_2, \partial_\mu \phi_1, \partial_\mu \phi_2$
3) Symmetry under $\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta$

$$\phi_2 \to -\phi_1 \sin\theta + \phi_2 \cos\theta$$

(like rotation in two dimensions)

$$\boxed{\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} \left(-\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i^2 \right)}$$

Arbitrary convention (except for sign)

So that Hamiltonian is bounded below

Euler-Lagrange Equations: $\partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \dfrac{\partial \mathcal{L}}{\partial \phi_i}$

$$\boxed{\partial_\mu \partial^\mu \phi_i = +m^2 \phi_i}$$

Conserved current: $\left.\dfrac{\partial \phi_1}{\partial \theta}\right|_{\theta=0} = \phi_2$

$$\left.\frac{\partial \phi_2}{\partial \theta}\right|_{\theta=0} = -\phi_1$$

Infinitessimal symmetry transformation:

$\phi_1 \to \phi_1 + \theta \phi_2$, $\quad \phi_2 \to \phi_2 - \theta \phi_1$

$$S \to S + \int d^4x \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right)$$

$$= S + \int d^4x \sum_i \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \delta\phi_i \right]$$

(using the E-L equations)

$$= S + \int d^4x \sum_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right]$$

$$= S + \int d^4x \, \theta \sum_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta}\bigg|_{\theta=0} \right]$$

(using $\delta\phi_i = \theta \frac{\partial \phi_i}{\partial \theta}\Big|_{\theta=0}$)

But under the symmetry transformation $\mathcal{L} \to \mathcal{L}$, $S \to S$.

So $\partial_\mu J^\mu = 0$, where $\boxed{J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta}\bigg|_{\theta=0}}$

More generally, a symmetry leaves S invariant,

infinitesimal symmetry parameter

$$\mathcal{L} \to \mathcal{L} + \theta \partial_\mu F^\mu \text{ for some } F^\mu(\phi_i(x), \partial_\mu \phi_i(x), x)$$

w/o using the E-L equations.

Then $\boxed{J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta}\bigg|_{\theta=0} - F^\mu}$

For our theory, $J^\mu = -(\partial^\mu \phi_1)\phi_2 + (\partial^\mu \phi_2)\phi_1$

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Interpretation of current conservation:

$$\partial_\mu J^\mu = 0$$

$\int d^3x$: $$\int_V d^3x \, \partial_0 J^0 = -\int_V d^3x \, \partial_i J^i$$

$$= -\int_{\partial V} d^2x \, n^i J^i$$

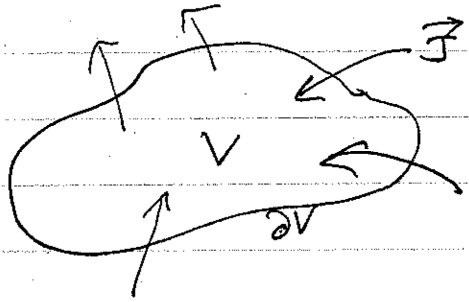
n^i = unit normal to boundary of V

Hence, $$\frac{d}{dt} \int_V d^3x \, J^0 = -\int_{\partial V} d^2x \, n^i J^i$$

Define charge $$\boxed{Q \equiv \int_V d^3x \, J^0}$$

Then $$\frac{dQ}{dt} = -\int_{\partial V} d^2x \, \hat{n} \cdot \vec{J}$$

ie, rate of change of charge in a volume V is the flux of current into V



In our example, $Q = \int d^3x \left(-\phi_2 \partial_0 \phi_1 + \phi_1 \partial_0 \phi_2 \right)$

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The Energy-Momentum Tensor

As an application of Noether's theorem in field theory, consider a theory of a scalar field $\phi(x)$ invariant under spacetime translations.

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$ for infinitessimal a^μ.

Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ transforms similarly:

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \left(\delta^\mu{}_\nu \mathcal{L}\right)$$

For each $\nu = 0,1,2,3$ there is a conserved current:

$$\boxed{T^\mu{}_\nu \equiv -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \delta^\mu{}_\nu\mathcal{L}}$$ Energy-Momentum Tensor

Conserved charge associated w/ time translations:

$$\boxed{H = \int d^3x\, T^{00} = \int d^3x \left[\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}}_{\Pi(x) = \text{Canonical momentum}}\partial_0\phi - \mathcal{L}\right]}$$ (Agrees w/ previously defined Hamiltonian)

Conserved charge associated w/ spatial translations:

$$P^i = \int d^3x\, T^{0i} = +\int d^3x\, T^0{}_i = -\int d^3x \left[\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_i\phi\right]$$, or

$$\boxed{P^i = -\int d^3x\, \Pi(x)\,\partial_i\phi(x)}$$ Spatial Momentum

Symmetry of the Energy-Momentum Tensor

In an arbitrary Lorentz-invariant theory, the energy-momentum tensor can be made symmetric in exchange of its indices

Given fields ϕ_a, $T^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - \eta^{\mu\nu}\mathcal{L}$

Add to $T^{\mu\nu}$ a tensor $\Delta T^{\mu\nu} = \partial_\lambda A^{\lambda\mu\nu}$ for some $A^{\lambda\mu\nu}$, antisymmetric in $\lambda \leftrightarrow \mu$.

$$\partial_\mu (T^{\mu\nu} + \Delta T^{\mu\nu}) = \partial_\mu T^{\mu\nu} + \cancelto{0}{\partial_\mu \partial_\lambda A^{\lambda\mu\nu}}$$
$$= \partial_\mu T^{\mu\nu}$$ by symmetry of mixed partial derivatives.

Hence, $T^{\mu\nu} + \Delta T^{\mu\nu}$ is conserved if $T^{\mu\nu}$ is conserved.

Under an infinitesimal Lorentz transformation, suppose

$$\phi_a \to \phi'_a = \sum_b S_{ab}\, \phi_b(\Lambda^{-1}x)$$

where $S_{ab} = \mathbb{1}_{ab} + \epsilon_{\mu\nu} \Sigma^{\mu\nu}_{ab}$ is a matrix rep. of the Lorentz group

antisymmetric in $\mu \leftrightarrow \nu$, $a \leftrightarrow b$

infinitesimal, antisymmetric parameters $\Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu - \epsilon^\mu{}_\nu$

Lorentz invariance of $\mathcal{L}$ implies

$$\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) = 0$$

where $\delta\phi_a = \sum_b \epsilon_{\mu\nu} (\Sigma^{\mu\nu})_{ab} \phi_b$

Using the E-L eqn,

$$\sum_a \left[(\partial_\mu \Pi^\mu_a) \delta\phi_a + \Pi^\mu_a \, \delta(\partial_\mu \phi_a) \right] = 0$$

where $\Pi^\mu_a = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}$.

Use $\delta\big(\partial_\mu \phi_a(\Lambda^{-1}x)\big) \approx \partial_\mu \delta\phi_a - \epsilon^\nu{}_\mu \partial_\nu \phi_a + \mathcal{O}(\epsilon^2)$

$$\rightarrow \sum_a \Big(\partial_\lambda (\Pi^\lambda_a \delta\phi_a) - \Pi^\mu_a \eta^{\nu\lambda} \epsilon_{\lambda\mu} \partial_\nu \phi_a\Big) = 0$$

$$\sum_{ab} \partial_\lambda \Big(\Pi^\lambda_a \epsilon_{\mu\nu} \Sigma^{\mu\nu}_{ab} \phi_b\Big) = \sum_a \Pi^\mu_a \epsilon_{\lambda\mu} \partial^\lambda \phi_a$$

(*) $\xrightarrow{\text{antisymmetrizing}} \sum_a \frac{1}{2}\Big(\Pi^\mu_a \partial^\nu \phi_a - \Pi^\nu_a \partial^\mu \phi_a\Big) = \sum_{ab} \partial_\lambda \Big(\Pi^\lambda_a \Sigma^{\nu\mu}_{ab} \phi_b\Big)$

Define $A^{\lambda\mu\nu} = \sum_{ab}\Big(\Pi^\lambda_a \Sigma^{\mu\nu}_{ab} - \Pi^\mu_a \Sigma^{\lambda\nu}_{ab} - \Pi^\nu_a \Sigma^{\lambda\mu}_{ab}\Big)\phi_b$

$$\boxed{\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda A^{\lambda\mu\nu}}$$

$$= \sum_a \Pi^\mu_a \partial^\nu \phi_a - \eta^{\mu\nu} \mathcal{L}$$
$$+ \sum_{ab} \partial_\lambda \Big(-\Pi^\lambda_a \Sigma^{\nu\mu}_{ab} \phi_b\Big) - \sum_{ab} \partial_\lambda \Big(\big(\Pi^\mu_a \Sigma^{\lambda\nu}_{ab} + \Pi^\nu_a \Sigma^{\lambda\mu}_{ab}\big)\phi_b\Big)$$

using (*)

$$\overset{\downarrow}{=} \frac{1}{2} \sum_a \Big(\Pi^\mu_a \partial^\nu \phi_a + \Pi^\nu_a \partial^\mu \phi_a\Big) - \eta^{\mu\nu} \mathcal{L}$$
$$- \sum_{ab} \partial_\lambda \Big(\big(\Pi^\mu_a \Sigma^{\lambda\nu}_{ab} + \Pi^\nu_a \Sigma^{\lambda\mu}_{ab}\big)\phi_b\Big) ,$$

which is symmetric in $\mu \leftrightarrow \nu$ and conserved.