cf. Weinberg Ch. 3

Consequences of the Equivalence Principle
(for particle motion)

In the absence of gravity or any forces, in any inertial frame particles move in straight lines in spacetime,

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0, \quad \text{where } d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

and ξ^α is an inertial coordinate system.

Check:

For $\alpha = 0$, $\frac{d^2 t}{d\tau^2} = 0 \rightarrow t = a\tau + b$

$\alpha = i$, $\frac{d^2\xi^i}{d\tau^2} = 0 \rightarrow \xi^i = A^i\tau + B^i = A^i\left(\frac{t-b}{a}\right) + B^i$

$$= \left(\frac{A^i}{a}\right)t + \left(B^i - \frac{bA^i}{a}\right)$$

This describes a particle moving w/ velocity $\left(\frac{A^i}{a}\right)$.

In the presence of gravity but no other forces, the equivalence principle implies that there is a freely falling coordinate system ξ^α in which the particle's equation of motion is identical to that above, i.e.

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0$$

Suppose we consider the ~~same~~ motion in an arbitrary coordinate system x^μ, so that $\xi^\alpha = \xi^\alpha(x^\mu)$

$$0 = \frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \quad \text{(chain rule)}$$

$$= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \underbrace{\frac{\partial}{\partial x^\nu}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\right)\frac{dx^\nu}{d\tau}}_{\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\right)}\frac{dx^\mu}{d\tau}$$

Multiply by $\frac{\partial x^\lambda}{\partial\xi^\alpha}$, use $\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \delta^\lambda_\mu$.

$$\Rightarrow \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\delta^\lambda{}_\mu\frac{d^2x^\mu}{d\tau^2} + \underbrace{\frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

where $\boxed{\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}}$ is called the <u>affine connection.</u>

The proper time may also be expressed in the new coordinate system:

$$d\tau^2 = -\eta_{\alpha\beta}\,d\xi^\alpha\,d\xi^\beta$$

$$= -\eta_{\alpha\beta}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\right)\left(\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu\right)$$

$$\equiv -g_{\mu\nu}\,dx^\mu dx^\nu$$

where $g_{\mu\nu}$ is the metric tensor

$$\boxed{g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}}$$

Note! Along the trajectory of a photon $d\tau^2 = 0$, but we can instead use $\sigma \equiv \xi^0$ to parametrize the trajectory. The equation of motion and vanishing proper time become (in the freely falling frame)

$$\frac{d^2\xi^\alpha}{d\sigma^2} = 0$$

$$0 = -\eta_{\alpha\beta} \frac{d\xi^\alpha}{d\sigma} \frac{d\xi^\beta}{d\sigma}$$

which, in a general coordinate system becomes

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0$$

$$0 = -g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}$$

where $\Gamma^\mu_{\nu\lambda}$ and $g_{\mu\nu}$ are as before.

The proper time between two events with a given infinitesimal coordinate separation is determined by the <u>metric tensor</u> $g_{\mu\nu}$. The motion of a particle in a gravitational field is determined by the <u>affine connection</u> $\Gamma^\lambda_{\mu\nu}$. There is, in fact, a relation between $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$.

Recall that $g_{\mu\nu} = \dfrac{\partial \xi^\alpha}{\partial x^\mu} \dfrac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$

Differentiating w.r.t. x^λ gives

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \underset{\Gamma^\rho_{\lambda\mu} \frac{\partial \xi^\alpha}{\partial x^\rho}}{\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu} \underset{\Gamma^\rho_{\lambda\nu} \frac{\partial \xi^\beta}{\partial x^\rho}}{\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}} \eta_{\alpha\beta}$$

$$= \Gamma^\rho_{\lambda\mu} \underbrace{\frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}}_{g_{\rho\nu}} + \Gamma^\rho_{\lambda\nu} \underbrace{\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \eta_{\alpha\beta}}_{g_{\rho\mu}}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\rho\mu}$$

It follows that (Exercise):

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2 g_{\kappa\nu} \Gamma^\kappa_{\lambda\mu}$$

Define $g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$, i.e.

$$\boxed{g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma}$$

From above: $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2 g_{\kappa\nu} \Gamma^\kappa_{\lambda\mu}$

Contract with $g^{\nu\sigma}$:

$$g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right) = 2 g_{\kappa\nu} g^{\nu\sigma} \Gamma^\kappa_{\lambda\mu}$$

$$= 2 \delta_\kappa{}^\sigma \Gamma^\kappa_{\lambda\mu}$$

$$= 2 \Gamma^\sigma_{\lambda\mu}$$

In other words,

$$\boxed{\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right)}$$

In terms of the metric $g_{\mu\nu}$, $\Gamma^\sigma_{\lambda\nu}$ is also called the <u>Christoffel symbol</u>.

<u>Consequences of the relation between $\Gamma^\sigma_{\lambda\mu}$ and $g_{\mu\nu}$:</u>

(1) The Eq. of motion of a freely falling particle automatically maintains the form of the proper time interval $d\tau$

$$\frac{d}{d\tau}\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\right) = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{dx^\lambda}{d\tau} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$+ g_{\mu\nu} \underbrace{\frac{d^2 x^\mu}{d\tau^2}}_{-\Gamma^\mu_{\kappa\lambda} \frac{dx^\kappa}{d\tau} \frac{dx^\lambda}{d\tau}} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \underbrace{\frac{d^2 x^\nu}{d\tau^2}}_{-\Gamma^\nu_{\kappa\lambda} \frac{dx^\kappa}{d\tau} \frac{dx^\lambda}{d\tau}}$$

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = \left[\frac{\partial g_{\kappa\sigma}}{\partial x^\lambda} - g_{\mu\sigma}\Gamma^\mu_{\kappa\lambda} - g_{\nu\kappa}\Gamma^\nu_{\sigma\lambda}\right] \times \frac{dx^\kappa}{d\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau}$$

The term in brackets vanishes by the relation between $\Gamma^\mu_{\kappa\lambda}$ and $g_{\mu\nu}$. (Exercise)

Hence $\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0$

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -C,$$ where C is a constant of the motion.

If we choose $C=1$ then $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$ everywhere along the trajectory.

Similarly, for a massless photon, $C=0$ as an initial condition, and $g_{\mu\nu}dx^\mu dx^\nu = 0$ along the trajectory.

(2) The law of motion of freely falling bodies satisfies a variational principle, namely that the proper time is stationarized.

Define $$T(A\to B) = \int_A^B \frac{d\tau}{dp}\,dp = \int_A^B \left\{-g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right\}^{1/2} dp,$$

where p is an arbitrary parameter along the trajectory, which begins at point A and ends at point B.

Now let $x^\mu(p) \to x^\mu(p) + \delta x^\mu(p)$ with $\delta x^\mu = 0$ at p_A, p_B

$$\delta T(A\to B) = \frac{1}{2}\int_A^B \underbrace{\left\{-g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right\}^{-1/2}}_{\frac{dp}{d\tau}} \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{dp}\frac{dx^\nu}{dp} - 2 g_{\mu\nu}\frac{d\delta x^\mu}{dp}\frac{dx^\nu}{dp}\right] dp$$

↑ from symmetry of $\mu \leftrightarrow \nu$

$$\delta T(A\to B) = -\int_A^B \left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\lambda\nu}\frac{d\delta x^\lambda}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau$$

↑ integrate by parts

$$= -\int_A^B \left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \left(\frac{\partial g_{\lambda\nu}}{\partial x^\sigma}\frac{dx^\sigma}{d\tau}\right)\frac{dx^\nu}{d\tau} - g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2}\right] \delta x^\lambda\, d\tau$$

$$= -\int_A^B \left\{\left[\frac{1}{2}\frac{\partial g_{\mu\sigma}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\sigma}\right]\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} - g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2}\right\}\delta x^\lambda\, d\tau$$

$$= -\int_A^B \left\{\left[\frac{1}{2}\underbrace{g_{\lambda\nu}g^{\nu\kappa}}_{\delta_\lambda^{\kappa}}\frac{\partial g_{\mu\sigma}}{\partial x^\kappa} - \underbrace{g_{\lambda\nu}g^{\nu\kappa}}_{\delta_\lambda^{\kappa}}\frac{\partial g_{\kappa\mu}}{\partial x^\sigma}\right]\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} - g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2}\right\}\delta x^\lambda\, d\tau$$

$$= -\int_A^B \left\{\frac{1}{2}g^{\nu\kappa}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\mu\kappa}}{\partial x^\sigma} - \frac{\partial g_{\sigma\kappa}}{\partial x^\mu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} - \frac{d^2x^\nu}{d\tau^2}\right\} \cdot g_{\lambda\nu}\delta x^\lambda\, d\tau$$

$$= \int_A^B \left\{\underbrace{\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau}}_{=0 \text{ along freely falling trajectory, i.e.}}\right\} g_{\lambda\nu}\delta x^\lambda\, d\tau$$

$$\boxed{\delta T(A\to B) = 0}$$

We will return to the implications of the stationarization of the proper time along trajectories shortly.

(3) Consider a slowly moving particle, $\frac{d\vec{x}}{d\tau}$ negligible compared to $\frac{dt}{d\tau}$, in a weak stationary gravitational field.

$$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$\approx \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2$$

(stationary field)

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\kappa}\left(\cancelto{0}{\frac{\partial g_{\kappa 0}}{\partial x^0}} + \cancelto{0}{\frac{\partial g_{\kappa 0}}{\partial x^0}} - \frac{\partial g_{00}}{\partial x^\kappa}\right)$$

$$= -\frac{1}{2}g^{\mu\kappa}\frac{\partial g_{00}}{\partial x^\kappa}$$

Suppose we adopt a nearly Cartesian coordinate system in the weak field, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$

To first order in $h_{\mu\nu}$: $\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\kappa}\frac{\partial h_{00}}{\partial x^\kappa}$

The equations of motion become:

$$\frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = \text{const.}$$

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{ij}\frac{\partial h_{00}}{\partial x^j}\left(\frac{dt}{d\tau}\right)^2 \Rightarrow \frac{d^2\vec{x}}{d\tau^2} = \frac{1}{2}\nabla h_{00}\left(\frac{dt}{d\tau}\right)^2$$

Divide the second eqn. by the constant $\left(\frac{dt}{d\tau}\right)^2$:

$$\boxed{\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\nabla h_{00}}$$

Comparing with Newtonian gravity,

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\phi \longrightarrow h_{00} = -2\phi + \text{constant}$$

ϕ: gravitational potential

choose such that $\phi \to 0$ at infinity, i.e. $\phi = -\frac{GM}{r}$

$$\Rightarrow g_{00} = -(1+2\phi)$$

At the surface of the earth, $|\phi| \sim 10^{-9}$

sun 10^{-6}

white dwarf 10^{-4}

→ The assumption $|h_{\mu\nu}| \ll 1$ is self consistent in typical physical situations

(4) Time Dilation, Gravitational Redshift

Consider a clock in a gravitational field, though not necessarily in free fall.

In a locally inertial frame the proper time between ticks is $\Delta\tau = (-\eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta)^{1/2}$

In an arbitrary coordinate system,

$$\Delta\tau = (-g_{\mu\nu}\, dx^\mu\, dx^\nu)^{1/2}$$

In the rest frame of the clock, with the time interval between clicks dt,

$$\Delta\tau = \sqrt{-g_{00}}\, dt, \text{ or } \quad \boxed{dt = \frac{\Delta\tau}{\sqrt{-g_{00}}}}$$

For a weak stationary field with $g_{00} = -(1+2\phi)$,

$$dt = \frac{\Delta\tau}{\sqrt{1+2\phi}} \approx \Delta\tau\,(1-\phi)$$ Time dilation

One can measure the time dilation by observing clocks from different points in space.
Suppose an atom emits light w/ some frequency ν_2 from pt. 2 as observed at pt. 1, so that the time between crests of the wave is $dt_2 = \frac{1}{\nu_2} = \Delta\tau / \sqrt{-g_{00}(x_2)}$.

If the same light is emitted at pt. 1 and observed also at pt. 1, then the time between crests is
$$dt_1 = \frac{1}{\nu_1} = \Delta\tau / \sqrt{-g_{00}(x_1)}$$

The ratio of the frequency of light from pt 2 observed at pt. 1, to that of light from pt. 1 observed at pt. 1 is

$$\boxed{\frac{\nu_2}{\nu_1} = \left(\frac{g_{00}(x_2)}{g_{00}(x_1)}\right)^{1/2}}$$

In the weak field limit, $\frac{\nu_2}{\nu_1} = 1 + \frac{\Delta\nu}{\nu}$, $g_{00} \approx -(1+2\phi)$

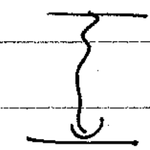
$$\rightarrow 1 + \frac{\Delta\nu}{\nu} \approx \left(\frac{1+2\phi(x_2)}{1+2\phi(x_1)}\right)^{1/2} \approx 1 + \phi(x_2) - \phi(x_1)$$

$$\boxed{\frac{\Delta\nu}{\nu} = \phi(x_2) - \phi(x_1)}$$ Gravitational Redshift

Example: Light from the sun is redshifted by 2 parts per million on the way to Earth.

Using the Mossbauer effect, Pound and Rebka measured the increase in frequency of 14.4keV light emitted by Fe^{57} falling 22.6m on Earth, with

$$\Delta\phi \approx -2.5 \times 10^{-15}$$



$$\frac{\Delta\nu}{\nu} \approx 2.5 \times 10^{-15}$$

The gravitational redshift plays an important role in astronomical observations of light from distant gravitational potential wells, in which case the redshift can be used to deduce the mass distribution at cosmological distances.