What is a Particle?

Particles in Minkowski Spacetime

Birrell & Davies Ch. 2

Consider a scalar field with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2.$$

The Euler-Lagrange equation is the Klein-Gordon equation

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = 0.$$

One set of solutions is

$$u_{\vec{k}}(t,\vec{x}) = \exp\left[i\vec{k}\cdot\vec{x} - i\omega t\right]\cdot\frac{1}{\sqrt{(2\pi)^3\cdot 2\omega}}$$

where

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}, \quad -\infty < k_i < \infty.$$

These modes are eigenfunctions of $i\frac{\partial}{\partial t}$,

$$i\frac{\partial}{\partial t}u_{\vec{k}}(t,\vec{x}) = \omega_{\vec{k}}u_{\vec{k}}(t,\vec{x}).$$

The eigenvalue $\omega_{\vec{k}}$ is called the (angular) frequency and $\vec{k}$ is called the momentum.

A generic solution to the Euler-Lagrange equation can be expanded in the modes $u_{\vec{k}}$:

$$\phi(t,\vec{x}) = \sum_{\vec{k}}\left[a_{\vec{k}}\,u_{\vec{k}}(t,\vec{x}) + a_{\vec{k}}^{\dagger}\,u_{\vec{k}}^{*}(t,\vec{x})\right]$$

Canonical Quantization of the scalar field proceeds by imposing commutation relations analogous to
$$\begin{cases} [x,p] = i\hbar \\ [x,x] = [p,p] = 0. \end{cases}$$

The canonical momentum conjugate to ϕ is defined by analogy with $p = \frac{\partial L}{\partial \dot{q}}$ in particle mechanics:
$$\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi$$

The canonical commutation relations then take the form
$$\begin{cases} [\phi(t,\vec{x}), \phi(t,\vec{x}')] = 0 \\ [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0 \\ [\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\,\delta^3(\vec{x}-\vec{x}') \end{cases}$$

By substituting the expansion for ϕ and π in terms of $u_{\vec{k}}(t,\vec{x})$, the canonical commutation relations become commutation relations for the coefficients $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$, which are now operators:
$$\begin{cases} [a_{\vec{k}}, a_{\vec{k}'}] = 0 \\ [a_{\vec{k}}^{\dagger}, a_{\vec{k}'}^{\dagger}] = 0 \\ [a_{\vec{k}}, a_{\vec{k}'}^{\dagger}] = \delta^3(\vec{k}-\vec{k}') \end{cases}$$
These resemble the harmonic oscillator commutators for annihilation and creation operators $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$, respectively.

The vacuum is defined by the condition
$$a_{\vec{k}}|0\rangle = 0 \quad \text{for all } \vec{k}.$$

The state $|1_{\vec{k}}\rangle \equiv a_{\vec{k}}^{\dagger}|0\rangle$ is an eigenstate of the number operator $N \equiv \sum_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}}$ with eigenvalue 1.

The Hamiltonian for ϕ is $H = \pi \partial_t \phi - \mathcal{L}$ and can be written in terms of $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$:

$$H = \frac{1}{2} \sum_{\vec{k}} \omega_{\vec{k}} \left(a_{\vec{k}}^{\dagger} a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^{\dagger} \right).$$

Similarly the momentum is

$$\vec{P} = \sum_{\vec{k}} \vec{k}\, a_{\vec{k}}^{\dagger} a_{\vec{k}}$$

The state $|1_{\vec{k}}\rangle$ is an eigenstate of H and $\vec{P}$ with eigenvalues

$$E = \omega_{\vec{k}} + \sum_{\vec{k}} \tfrac{1}{2}\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} + \underbrace{\sum_{\vec{k}} \tfrac{1}{2}\omega_{\vec{k}}}_{\text{zero-point energy.}}$$

$$\vec{P} = \vec{k}$$

The constant $\sum_{\vec{k}} \frac{1}{2}\omega_{\vec{k}}$ is the zero-point energy and contributes to every state the same amount.

Hence $|1_{\vec{k}}\rangle$ has the quantum numbers of a 1-particle state with momentum $\vec{k}$ and energy $\omega_{\vec{k}}$.

We call $|1_{\vec{k}}\rangle$ a 1-particle state.

Acting on the vacuum by additional creation operators $a^{+}_{\vec{k}_i}$ gives multiparticle states labeled by $\{\vec{k}_i\}$.

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Particles in curved spacetime

The notion of a particle is more subtle in a generic background spacetime. Which coordinate system should be used to define the frequency (what is $i\frac{\partial}{\partial t}$?) and to choose the complete set of modes $\{u_j(t,\vec{x})\}$?

Consider the Lagrangian density

$$\mathcal{L} = \sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2\right)$$

The Euler-Lagrange equation for ϕ is

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi = 0$$

Suppose we find a complete set of modes $\{u_j, u_j^*\}$ and decompose ϕ as before:

$$\phi(t,\vec{x}) = \sum_j\left[a_j u_j(t,\vec{x}) + a_j^{+} u_j^*(t,\vec{x})\right]$$

with appropriately normalized $u_j(t,\vec{x})$, we impose the canonical commutation relations

$$[a_i, a_j^{+}] = \delta_{ij}\ , \text{etc.}$$

The vacuum satisfies $a_i|0\rangle = 0$ for all i.

In Minkowski space the vacuum is invariant under Poincaré transformations. A generic spacetime does not have Poincaré invariance, so there is no reason to expect the vacuum to have any such invariance.

So, consider a second complete set of modes $\{v_j, v_j^*\}$ and decompose the field ϕ as

$$\phi(t,\vec{x}) = \sum_j \left[b_j v_j(t,\vec{x}) + b_j^+ v_j^*(t,\vec{x}) \right]$$

The vacuum defined by $b_j|\bar{0}\rangle = 0$ for all j in general does not coincide with the vacuum $|0\rangle$ defined by $a_j|0\rangle = 0$.

Since $\{u_j\}$ and $\{v_j\}$ are both complete sets, we can write

$$\left.\begin{aligned} v_j &= \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right) \\ u_i &= \sum_j \left(\alpha_{ji}^* v_j - \beta_{ji} v_j^* \right) \end{aligned}\right\} \text{Bogdubov transformation}$$

The creation and annihilation operators can also be related to one another in the two bases.

$$a_i = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^+ \right)$$

$$b_j = \sum_i \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^+ \right)$$

The Bogoliubov coefficients α_{ij}, β_{ij} satisfy the following:

$$\left.\begin{aligned} &\sum_k \left(\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*\right) = \delta_{ij} \\ &\sum_k \left(\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}\right) = 0 \end{aligned}\right\}$$

To preserve canonical commutation relations on b_i, b_i^+.

As long as $\beta_{ji} \neq 0$ the vacua and excitations of the vacua in the two bases disagree.

$$a_i |\bar{0}\rangle = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^+\right) |\bar{0}\rangle$$

$$= \beta_{ji}^* |\bar{1}_j\rangle \neq 0$$

If $N_i = a_i^+ a_i$ is the number operator for the number of u_i-mode particles, then the expectation value for the number of u_i-mode particles in the state $|\bar{0}\rangle$ is

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2 .$$

$\Rightarrow$ The vacuum of the v_j modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode.

What Does a Particle Detector Detect?

Unruh and DeWitt modeled a particle detector as an idealized point particle with internal energy levels labeled by E, coupled to the scalar field ϕ by an interaction

$$\mathcal{L} = c\, m(\tau)\, \phi[x^\mu(\tau)]\, \delta^3(\vec{x} - \vec{x}(\tau))$$

where $x^\mu(\tau)$ describes the trajectory of the point particle and τ is the particle's proper time. $m(\tau)$ = monopole operator

Suppose the field ϕ is in the Minkowski vacuum $|0_M\rangle$, in the Minkowski spacetime. Depending on the trajectory $x^\mu(\tau)$, it is possible that the detector will transition to an excited state with energy $E > E_0$ (E_0 = ground state energy), while the field transitions to a state $|\Psi\rangle$.

Perturbation theory: transition amplitude a is

$$a = ic \langle E, \Psi | \int_{-\infty}^{\infty} d\tau\, m(\tau)\, \phi[x^\mu(\tau)]\, | 0_M, E_0 \rangle$$

Heisenberg picture evolution of $m(\tau)$: $m(\tau) = e^{iH_0\tau}\, m(0)\, e^{-iH_0\tau}$

$H_0 |E\rangle = E|E\rangle$

$$a = ic \langle E | m(0) | E_0 \rangle \int_{-\infty}^{\infty} d\tau\, e^{i(E-E_0)\tau} \langle \Psi | \phi(x) | 0_M \rangle$$

$a \neq 0$, if $|\Psi\rangle = |1_{\vec{k}}\rangle$:
(possibly)

$$\langle 1_{\vec{k}} | \phi(x) | 0_M \rangle = \frac{1}{\sqrt{16\pi^3 \omega_{\vec{k}}}} \exp\left(-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t\right)$$

$$\Rightarrow a_{\vec{k}} = \frac{ic\langle E|m(0)|E_0\rangle}{\sqrt{16\pi^3 \omega_{\vec{k}}}} \int_{-\infty}^{\infty} e^{i(E-E_0)\tau} e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t}$$

Example: Inertial particle detector

$$\vec{x} = \vec{x}_0 + \vec{v}t = \vec{x}_0 + \frac{\vec{v}\tau}{\sqrt{(1-v^2)}}$$

$$a_{\vec{k}} = \frac{ic\langle E|m(0)|E_0\rangle}{\sqrt{16\pi^3 \omega_{\vec{k}}}} \cdot 2\pi\delta\left(E - E_0 + \frac{(\omega - \vec{k}\cdot\vec{v})}{\sqrt{1-v^2}}\right)$$

But $E > E_0$ and $\vec{k}\cdot\vec{v} \leq |\vec{k}||\vec{v}| < \omega$

$\Rightarrow E - E_0 + \frac{(\omega - \vec{k}\cdot\vec{v})}{\sqrt{1-v^2}} > 0 \longrightarrow \delta$-fn vanishes

$\rightarrow a_{\vec{k}} = 0$

☆ $\Rightarrow$ An inertial particle detector in the Minkowski spacetime vacuum detects no particles.

Example: Noninertial detector trajectory

Transition probability to all possible E and ψ

$$P = \sum_E |a_{\psi,E}|^2 = c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \, \mathcal{F}(E-E_0)$$

where $$\mathcal{F}(E) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \, e^{-iE(\tau-\tau')} \underbrace{\langle 0_M|\phi(x^\mu(\tau))\,\phi(x^\mu(\tau'))|0_M\rangle}_{G^+(x(\tau),\, x(\tau'))}$$

= positive-frequency Wightman function

$\mathcal{F}(E)$ = detector response function
— represents the bath of particles detected by the detector as a result of its motion.

$c^2|\langle E|m(0)|E_0\rangle|^2$ factor describes selectivity of detector to the bath — depends on details of detector.

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Example: massless field ϕ, $m^2=0$, $z=(t^2+\alpha^2)^{1/2}$ trajectory, constant α^{-1} = acceleration
$G^+(x,x')$ can be calculated, then $\mathcal{F}(E)$ determined.

Result:
$$P = \frac{c^2}{2\pi} \sum_E \frac{(E-E_0)\,|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)\alpha}-1}$$

This is equivalent to what the particle detector would have detected if unaccelerated but in a bath of particles at temperature $T = \frac{1}{2\pi\alpha k_B} = \frac{\text{acceleration}}{2\pi k_B}$

Boltzmann's const

* This is the main result, known as the Unruh effect: An accelerated particle detector in Minkowski spacetime (Rindler spacetime) detects a finite-temperature distribution of particles with temperature proportional to the acceleration.

Another way to understand this result, which we will not explain here, is in terms of the particle horizon of the accelerated observer. (Recall from our earlier discussion of Rindler spacetime that Rindler coordinates cover only part of Minkowski spacetime.)

A related result is that an observer far from a black hole observes a finite-temperature distribution of particles with temperature $T_H = \frac{1}{8\pi G M k_B}$

Hawking temperature

(Temperatures in units where $\hbar = c = 1$. For example,

$$T_H = \frac{\hbar c^3}{8\pi G M k_B}.$$
)

Another related result is that particles can be created in a non-static spacetime.

To Summarize: Gravity + Quantum = Interesting!