The Multicomponent Universe

We will consider a more general universe containing several perfect fluids with equations of state approximately given by $p_i = w_i \rho_i$, where w_i is called the equation of state parameter of the fluid component labeled by i.

There is a separate conservation equation for each fluid component:

$$\frac{d}{dR}(\rho_i R^3) = -3 p_i R^2 ,$$ which can be written

$$\frac{d\rho_i}{dR} + 3(p_i + \rho_i) \cdot \frac{1}{R} = 0$$

Multiplying by $\frac{dR}{dt}$,

$$\boxed{\frac{d\rho_i}{dt} + 3(p_i + \rho_i) H = 0}, \text{ where } H = \frac{\dot{R}}{R}.$$

With $p_i = w_i \rho_i$, $\quad \frac{d\rho_i}{dt} + 3(1+w_i)\rho_i \frac{\dot{R}}{R} = 0$

The redshift parameter is $\boxed{z = \frac{R_0}{R} - 1}$

$$\frac{dz}{dt} = -\frac{R_0}{R^2}\dot{R}$$

$$\frac{d\rho_i}{dt} \cdot \frac{dt}{dz} + 3(1+w_i)\rho_i \frac{\dot{R}}{R}\left(-\frac{R^2}{R_0 \dot{R}}\right) = 0$$

$$\frac{d\rho_i}{dz} - 3(1+w_i)\rho_i (1+z)^{-1} = 0$$

Solution: $\boxed{\rho_i = \rho_{0i} (1+z)^{3(1+w_i)}}$

The Friedmann Equations can be written

$$\boxed{H^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{R^2}}$$

Defining $\Omega_i \equiv \frac{\rho_i}{\rho_c} = \frac{8\pi G \rho_i}{3H^2}$, $\Omega_{curv} \equiv -\frac{k}{R^2 H^2}$,

the Friedmann eqn. takes the form

$$\boxed{\sum_i \Omega_i + \Omega_{curv} = 1}$$

Given the solution for $\rho_i(z)$,

$$\boxed{H^2(z) = H_0^{\ 2}(1+z)^2\Big(1+\sum_i \underbrace{\Omega_i}_{\text{today}}\big[(1+z)^{1+3w_i} - 1\big]\Big)}$$

This formula for the Hubble constant is useful for constraining the properties of the matter content of the universe in terms of Ω_i and w_i.

Imagine a distant light source with absolute luminosity L (energy/time). The light is detected at $r=0$ with flux F (energy/time/area) and redshift z_s ← source.

The proper area of a sphere at coordinate radius r_s is $4\pi r_s^2 R_0^2$.

The energy of each emitted photon is reduced by the factor $(1+z_s)^{-1}$, and the time interval between photons received is increased by another factor of $(1+z_s)^{-1}$.

Hence, $\boxed{L = 4\pi r_s^2 R_0^2 (1+z_s)^2 F}$

The <u>luminosity distance</u> d_L is defined so that

$L = 4\pi d_L^2 F$, so that

$$\boxed{d_L^2 = r_s^2 R_0^2 (1+z_s)^2}$$ in an FRW universe.

Measurement of F and z_s with knowledge of L allows for a determination of $r_s(z)$, or alternatively $d_L(z)$. Such a measurement can be compared with the prediction for $r_s(z)$ given a set of Ω_i, w_i:

The light travels along a trajectory with $ds^2=0$

$$\rightarrow \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_s}^{t_0} \frac{dt}{R(t)} = \int_0^{z_s} \frac{dz}{R_0 H(z)}$$

For example, assuming $k=0$,

$$\boxed{r_s(z_s) = \frac{1}{R_0 H_0} \int_0^{z_s} \frac{dz}{(1+z)\sqrt{1+\sum_i \Omega_i \left[(1+z)^{1+3w_i} - 1\right]}}}$$

Type Ia supernovae have known intrinsic luminosities, so they are a good tool for observing the Hubble curve, which is often given as magnitude vs. redshift, where the magnitude m of a bright object is defined by

$$m = m_0 + 5\log_{10}(H_0 d_L)$$

↖ fiducial source magnitude

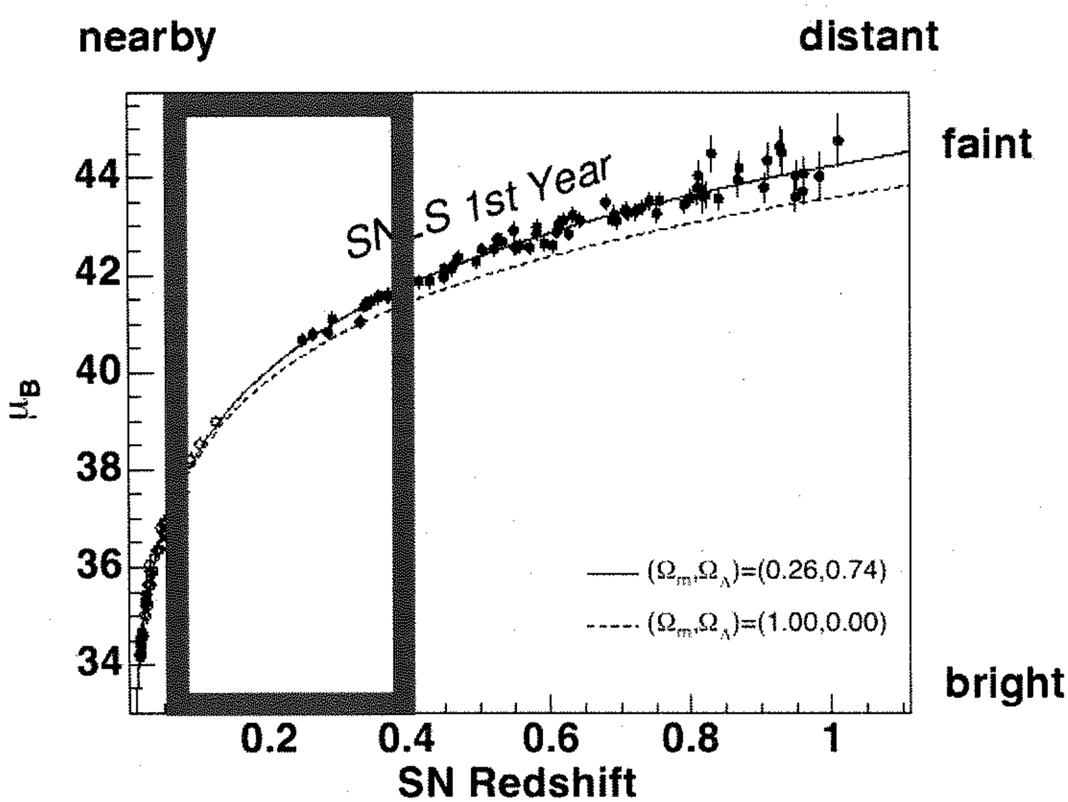
Other important sources of cosmological data include the power spectrum of the Cosmic Microwave Background which so far agrees with the predictions of inflation; measurements of the distribution of galaxies and clusters; gravitational lensing surveys; galactic rotation curves; ...

The resulting picture is that the universe is composed of 23% dark matter (by energy density)
73% dark energy
4% ordinary "baryonic" matter.

Dark matter ~ pressureless fluid $w=0$, $\Omega_M = 0.23$
Dark energy ~ cosmological constant $w=-1$, $\Omega_\Lambda = 0.73$
Baryonic matter ~ intergalactic gas, stars, etc. $\Omega_B = 0.04$

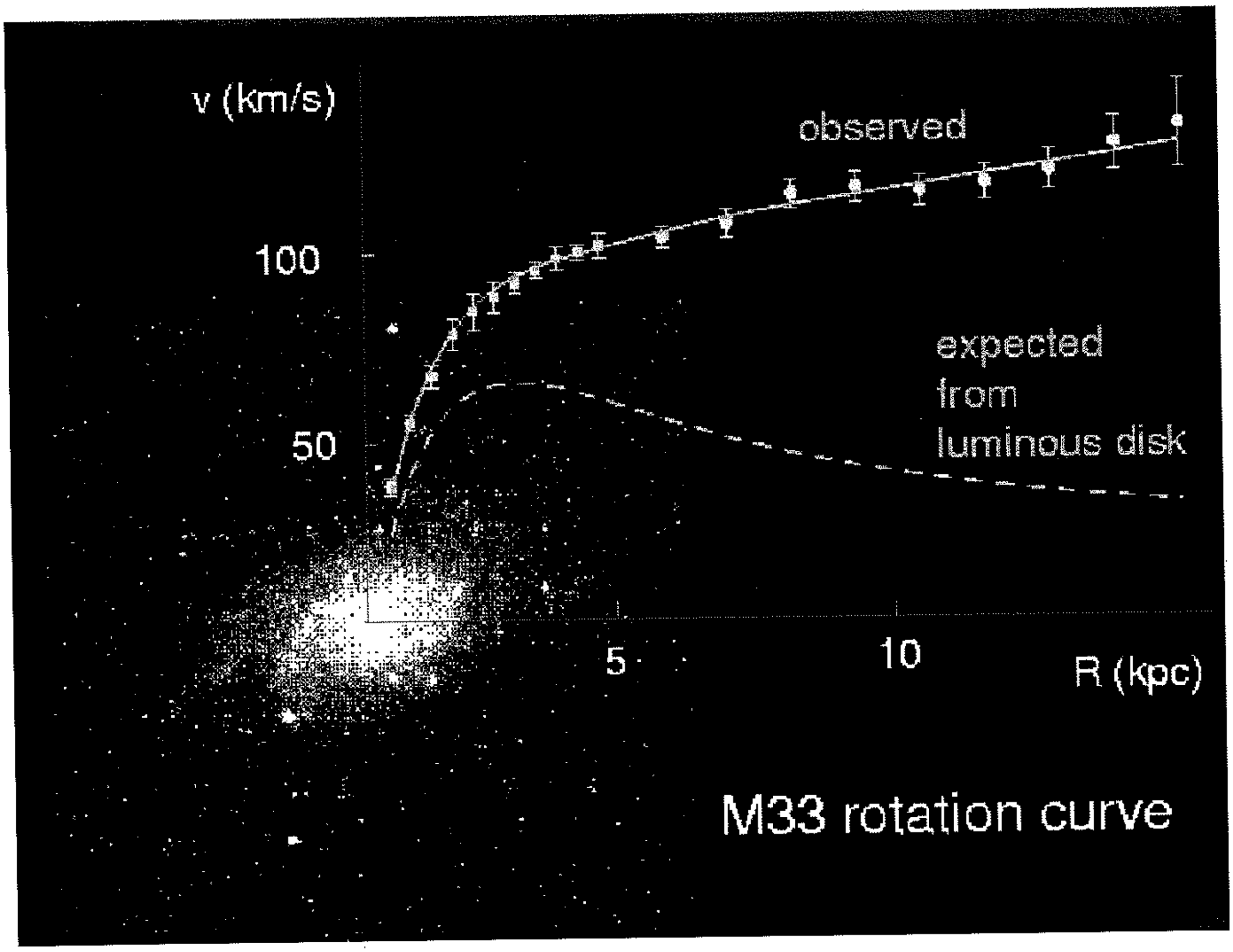
Age of Universe = 13.7 billion years
$H_0 = 70$ km/s/Mpc

Type Ia Supernovae



spiff.rit.edu/richmond/sdss/sn-survey/sn-survey.html

Galactic Rotation Curves

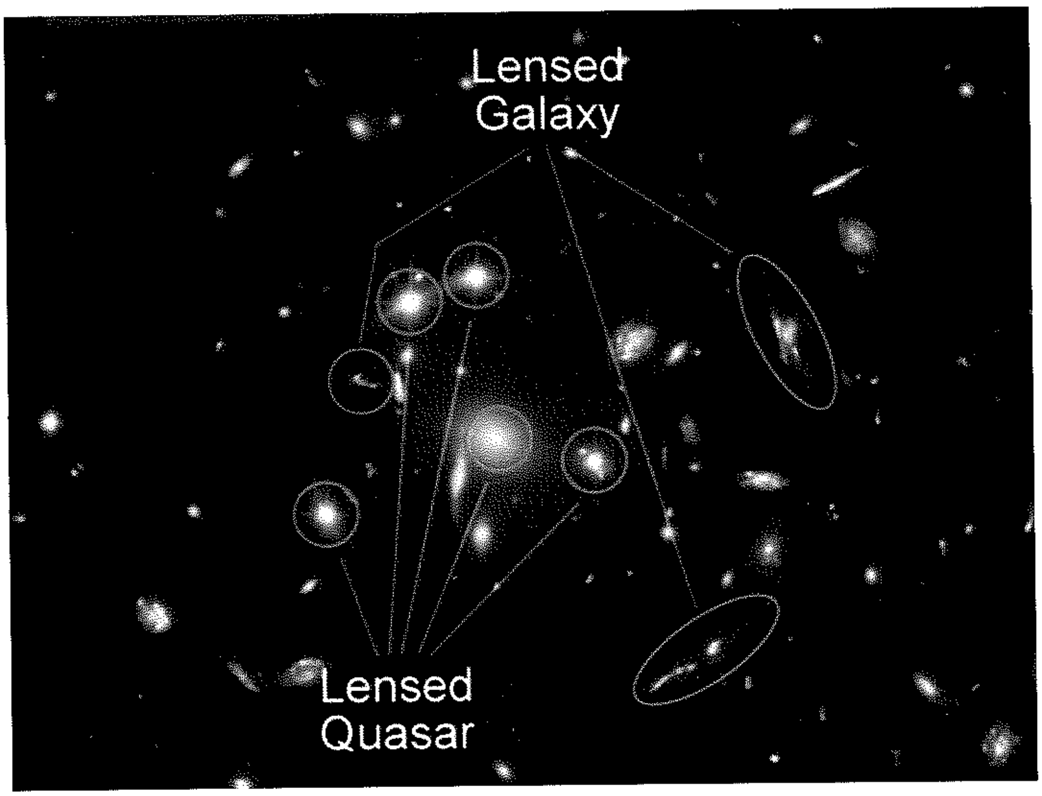


www.hep.shef.ac.uk/research/dm/intro.php

Bullet Cluster

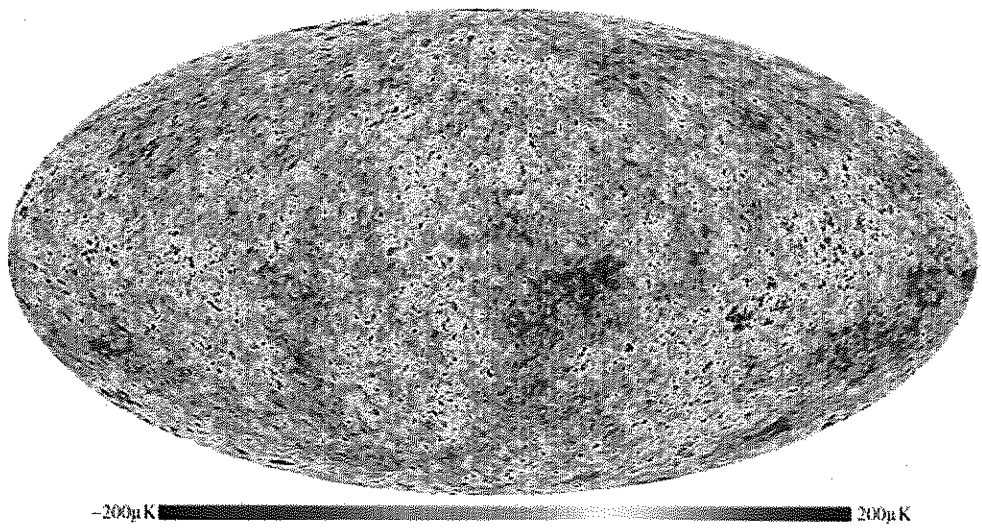
NASA

Gravitational Lensing

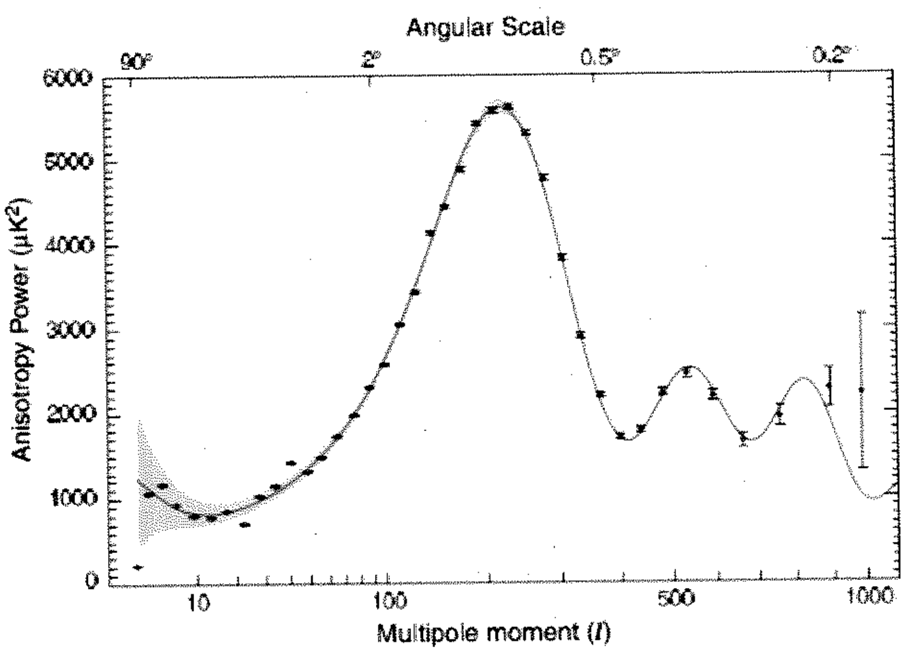


NASA

Cosmic Microwave Background



NASA/WMAP



NASA/WMAP

Weinberg 12.4

An Action Principle for Gravity

The gravitational action should be invariant under coordinate transformations and should be composed of terms with two derivatives of the metric and its inverse.

The invariant volume element is $\sqrt{|g|}\, d^4x$, and the only scalar that suits the bill is R. Hence, we will study the equations that follow by varying the action $S = S_M + S_G$, with S_M the matter action, and

$$\boxed{S_G = -\frac{1}{16\pi G}\int d^4x \sqrt{|g|}\, R}$$

Taking $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, using $R = g^{\mu\nu} R_{\mu\nu}$,

$$\delta\left(\sqrt{|g|}\, R\right) = \sqrt{|g|}\, R_{\mu\nu}\, \delta g^{\mu\nu} + R\, \delta\sqrt{|g|} + \sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu}.$$

From the definition of $R_{\mu\nu}$ in terms of $\Gamma^\lambda_{\mu\nu}$, and $\Gamma^\lambda_{\mu\nu}$ in terms of $g_{\mu\nu}$, it is straightforward to show the Palatini identity:

$$\boxed{\delta R_{\mu\nu} = \left(\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}}$$

Hence $\sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu} = \sqrt{|g|}\left[\left(g^{\mu\nu}\,\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(g^{\mu\nu}\,\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}\right]$

Using $V^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\, \partial_\mu\left(\sqrt{|g|}\, V^\mu\right)$,

$$\sqrt{|g|}\, g^{\mu\nu}\delta R_{\mu\nu} = \partial_\nu\left(\sqrt{|g|}\, g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right) - \partial_\lambda\left(\sqrt{|g|}\, g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right)$$

Hence, $\int d^4x \sqrt{|g|}\, g^{\mu\nu}\delta R_{\mu\nu} = 0.$

Using $\delta \ln\det M = \mathrm{Tr}\, M^{-1}\delta M$ with $M = g_{\mu\nu}$,

$$\boxed{\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\, g^{\mu\nu}\delta g_{\mu\nu}}$$

Using $\delta(g^{\mu\nu}g_{\nu\lambda}) = 0$, $\boxed{\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}}$

Hence, $\boxed{\delta S_G = \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]\delta g_{\mu\nu}}$

The energy-momentum tensor for matter may be defined in terms of the variation of the matter action with respect to the metric:

$$\boxed{\delta S_M = \frac{1}{2}\int d^4x \sqrt{|g|}\, T^{\mu\nu}\delta g_{\mu\nu}}$$

Hence, $\delta S = \delta S_G + \delta S_M$
$$= \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\,\delta g_{\mu\nu}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 8\pi G T^{\mu\nu}\right]$$

$\delta S = 0$ gives the equation of motion, which we recognize as the Einstein equation:

$$\boxed{R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G T^{\mu\nu}}$$