

The Expanding Universe

By modeling the universe as a perfect fluid we can use Einstein's eqs. to determine the evolution of the scale factor of the universe $R(t)$, and $K = +1, -1, \text{ or } 0$.

$$ds^2 = -dt^2 + R^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Nonvanishing components of the affine connection:

$$\Gamma_{rr}^t = \frac{R\dot{R}}{1-kr^2}, \quad \Gamma_{\theta\theta}^t = R\dot{R}r^2, \quad \Gamma_{\phi\phi}^t = R\dot{R}r^2\sin^2\theta$$

$$\Gamma_{tr}^r = \Gamma_{t\theta}^\theta = \Gamma_{t\phi}^\phi = \frac{\dot{R}}{R}$$

Defining $g_{ij} = R^2(t) \tilde{g}_{ij}$ for $i, j \in \{r, \theta, \phi\}$

$$\Gamma_{jk}^i = \frac{1}{2} (\tilde{g}^{-1})^{il} \left(\frac{\partial \tilde{g}_{lj}}{\partial x^k} + \frac{\partial \tilde{g}_{lk}}{\partial x^j} - \frac{\partial \tilde{g}_{ik}}{\partial x^l} \right) \equiv \tilde{\Gamma}_{jk}^l$$

↑
determined from 3D metric \tilde{g}_{ij}

Ricci tensor: $R_{tt} = \frac{3\ddot{R}}{R}$

$$R_{ti} = 0$$

$$R_{ij} = \tilde{R}_{ij} - (R\ddot{R} + 2\dot{R}^2) \tilde{g}_{ij}$$

where

$$\begin{aligned} \tilde{R}_{ij} &= \partial_j \tilde{\Gamma}_{ki}^k - \partial_k \tilde{\Gamma}_{ij}^k + \tilde{\Gamma}_{li}^k \tilde{\Gamma}_{kj}^l - \tilde{\Gamma}_{ij}^k \tilde{\Gamma}_{kl}^l \\ &= -2K \tilde{g}_{ij} \end{aligned}$$

$$\Rightarrow R_{ij} = -(R\ddot{R} + 2\dot{R}^2 + 2k) \tilde{g}_{ij}$$

Einstein Eqs: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$

Trace: $-R = -8\pi G T^\mu{}_\mu$

$$\Rightarrow R_{\mu\nu} = -8\pi G (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu} T^\mu{}_\mu)$$

$$\equiv -8\pi G S_{\mu\nu}$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}g_{\mu\nu} T$

Perfect Fluid: $T_{\mu\nu} = \rho g_{\mu\nu} + (\rho + p) U_\mu U_\nu$

where $U^\mu = (1, 0, 0, 0)$ — fluid elements are at rest in comoving coordinates.

Assume ρ, p depend only on t (homogeneous, isotropic)

$$S_{\mu\nu} = \frac{1}{2}(\rho - p)g_{\mu\nu} + (\rho + p)U_\mu U_\nu$$

$$\Rightarrow \begin{cases} S_{tt} = \frac{1}{2}(\rho + 3p) \\ S_{it} = 0 \\ S_{ij} = \frac{1}{2}(\rho - p)R^2 \tilde{g}_{ij} \end{cases}$$

$$R_{tt} = -8\pi G S_{tt} \Rightarrow 3\ddot{R} = -4\pi G(\rho + 3p)R$$

$$R_{ij} = -8\pi G S_{ij} \Rightarrow R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi G(\rho - p)R^2$$

$$\Rightarrow \frac{1}{3}R_{ij} - \frac{1}{6}R'R_{tt} = \ddot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$

↑ scale factor not curvature scalar

Conservation of Energy-Momentum:

$$0 = D_\nu T^{\mu\nu} \\ = \partial_\nu g^{\mu\nu} \frac{1}{\sqrt{g}} \partial_\nu [\sqrt{g} (\rho + p) U^\mu U^\nu] + \Gamma^\mu_{\nu\lambda} (\rho + p) U^\nu U^\lambda$$

$$\mu=t: -\dot{\rho} + \frac{\sqrt{1-kr^2}}{r^2 \sin\theta} \frac{r^2 \sin\theta}{\sqrt{1-kr^2}} \frac{d}{dt} [R^3 (\rho + p)] + 0 = 0$$

$$\Rightarrow \dot{\rho} R^3 = \frac{d}{dt} [R^3 (\rho + p)]$$

$$-\rho \frac{d}{dt} (R^3) = \frac{d}{dt} (R^3 p) \\ = -\rho - 3R^2 \frac{dR}{dt}$$

$$\Rightarrow \boxed{\frac{d}{dR} (p R^3) = -3p R^2}$$

Given eqn. of state $p = p(\rho)$, this equation determines ρ as a function of R .

Example: pressureless dust $p=0$ } Matter-dominated
 $\Rightarrow \rho \propto R^{-3}$ (after radiation domination)

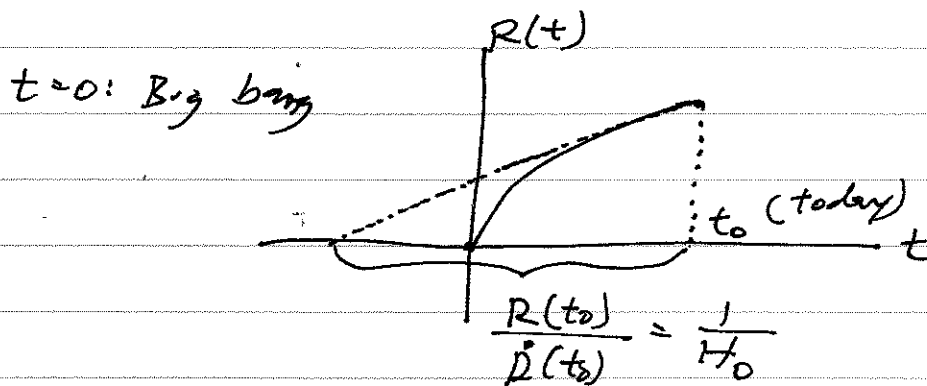
Example: massless photons $p = \frac{\rho}{3}$ } Radiation-Dominated
 $\rho \propto R^{-4}$ (Early universe)

tt component of Einstein's equations:

$$3\ddot{R} = -4\pi G(\rho + 3p)R$$

⇒ As long as $(\rho + 3p) > 0$, $\frac{\ddot{R}}{R} < 0$

→ Expansion rate of the universe decelerates in this model.



"Age of universe" $t_0 < \frac{1}{H_0}$ in this model.

$$D_\nu T^{\mu\nu} = 0 \rightarrow \frac{d}{dR}(\rho R^3) = -3pR^2$$

$$p > 0 \rightarrow \frac{d}{dR}(\rho R^3) < 0$$

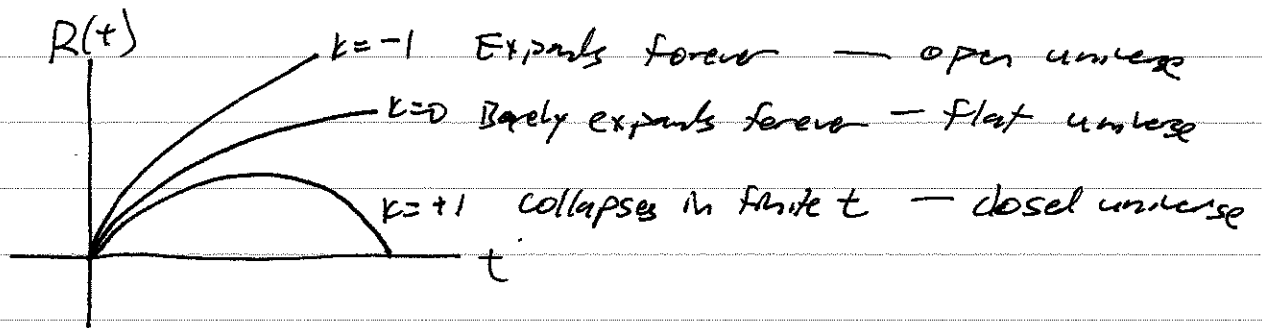
⇒ ρ decreases with R at least as fast as R^{-3} .

$$\text{Large } R: \dot{R}^2 + K = \frac{8\pi G}{3}\rho R^2 \rightarrow 0$$

$$K = -1: \dot{R}^2 \rightarrow 1 \text{ at } \infty: R(t) \rightarrow t \text{ as } t \rightarrow \infty$$

$$K = 0: \dot{R}^2 \rightarrow 0 \text{ at } \infty$$

$$K = +1: \dot{R}^2 \rightarrow 0 \text{ at } \rho R^2 = \frac{3}{8\pi G}, \text{ then } \dot{R} < 0 \text{ for larger } R$$



$k = 0$ is the borderline case between an ever-expanding universe and an ^{eventually} collapsing universe in this model.

$$\dot{R}^2 + k = \frac{8\pi G}{3} \rho R^2$$

$$k = 0 \Leftrightarrow \rho = \frac{3}{8\pi G} \frac{\dot{R}^2}{R^2} = \boxed{\frac{3H^2}{8\pi G} \equiv \rho_c} \quad \text{critical density}$$

$$k = +1 \Leftrightarrow \rho > \rho_c$$

$$k = -1 \Leftrightarrow \rho < \rho_c$$

Matter-Dominated Universe: $\rho = \rho_0 \frac{R_0^3}{R^3}$

$$\dot{R}^2 + k = \underbrace{\frac{8\pi G}{3} \rho_0 R_0^3}_{C} \frac{R^2}{R^3} = \frac{C}{R}$$

$$k = +1: \dot{R}^2 + 1 = \frac{C}{R} = \frac{R_{\max}}{R} \quad (\dot{R} = 0 \rightarrow C = R_{\max})$$

$$\text{Solutions: } \begin{cases} R = \frac{C}{2} (1 - \cos \eta) = \frac{R_{\max}}{2} (1 - \cos \eta) \\ t = \frac{R_{\max}}{2} (\eta - \sin \eta) \end{cases}, \quad 0 \leq \eta \leq 2\pi$$

Total lifetime of universe: $R = 0 \rightarrow \eta = 2\pi$

$$\rightarrow \boxed{t = \pi R_{\max}}$$

$$K=-1: \dot{R}^2 - 1 = \frac{C}{R}$$

$$\text{Solution: } \begin{cases} R = \frac{C}{2} (\cosh \eta - 1) \\ t = \frac{C}{2} (\sinh \eta - \eta) \end{cases}, \eta > 0$$

Large t : $R \sim t$

$$\text{Small } R: \dot{R}^2 \sim \frac{C}{R} \Rightarrow \sqrt{R} dR \sim C^{1/2} dt$$
$$\frac{2}{3} R^{3/2} \sim \sqrt{C} t$$
$$R \sim \left(\frac{3}{2}\right)^{2/3} C^{1/2} t^{2/3}$$

Small- t behavior independent of K .

$$K=0: R = \left(\frac{3}{2}\right)^{2/3} C^{1/2} t^{2/3}$$

$$\text{where } C = \frac{8\pi G}{3} \rho_0 R_0^3$$