Lorentz Invariance, Tensors, and Tensor Fields

We would like to explore the consequences of the equivalence principle and the general principle of relativity, namely that the laws of physics should be invariant under coordinate transformations of any sort.

First we will review what we mean by Lorentz invariance.

Begin with Rotations = Group of linear transformations of coordinates w/ one point fixed, such that length elements $ds^2 = d\vec{x}^2 = \sum_{i,j=1}^{3} dx^i \delta_{ij} dx^j$ are invariant.

Suppose the rotation is described by a matrix $R: x^i \to \sum_j R^i{}_j x^j$

Then $ds^2 \to \sum_{i,j,k,l=1}^{3} (R^i{}_k dx^k) \delta_{ij} (R^j{}_l dx^l)$

$$= \sum dx^k (R^T)_k{}^i \delta_{ij} R^j{}_l dx^l$$

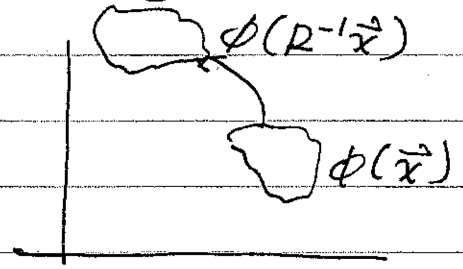
ds^2 will be invariant if $\boxed{R^T \mathbb{1} R = \mathbb{1}}$

Rotations also satisfy $\det R = +1$.

Such matrices R are called special orthogonal matrices, and form a representation of the rotation group $SO(3)$.

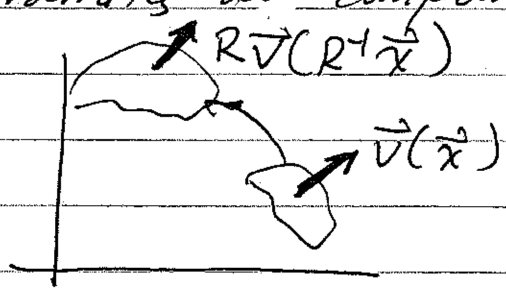
Next consider physical fields and how they transform under rotations.

A <u>scalar field</u> under rotations, $\phi(\vec{x})$ is a field which is invariant except for the relabeling of coordinates. We will take the <u>active</u> viewpoint in which the system is rotated (rather than the coordinate axes).



$$\phi(\vec{x}) \to \phi'(\vec{x}) = \phi(R^{-1}\vec{x})$$

A <u>vector field</u> $\vec{V}(\vec{x})$ rotates like the coordinates do componentwise.



$$\vec{V}(\vec{x}) \to R\vec{V}(R^{-1}\vec{x})$$

A <u>rank-n tensor field</u> transforms with n factors of the rotation matrix:

$$\boxed{T^{i_1 \cdots i_n}(\vec{x}) \to R^{i_1}{}_{j_1} R^{i_2}{}_{j_2} \cdots R^{i_n}{}_{j_n} T^{j_1 \cdots j_n}(R^{-1}\vec{x})}$$

An equation which is <u>rotationally covariant</u> transforms consistently under rotations.

An easy way to test covariance is by comparing the rotation indices throughout the equation.

$V^i(\vec{x}) = \phi(\vec{x})$ has an index on the left but not on the right $\rightarrow$ not covariant!

However, $\vec{V}(\vec{x}) \cdot \vec{V}(\vec{x}) = \phi(\vec{x})$ is covariant because both sides are rotational scalars.

In terms of the components of $\vec{V}(\vec{x})$, we can write

$$\vec{V} \cdot \vec{V} = \sum_{i=1}^{3} V^i V^i = \sum_{i,j=1}^{3} V^i \delta_{ij} V^j$$

☆ Summing over pairs of repeated indices leaves a tensor that transforms as though those indices were not there.

For example, $\sum_{i=1}^{3} V^i V^i \xrightarrow{R} \sum_{i,j,k=1}^{3} (R^i{}_j V^j)(R^i{}_k V^k)$

$$= \sum_{i,j,k=1}^{3} V^j (R^T)_j{}^i R^i{}_k V^k$$

$$= \sum_{j,k=1}^{3} V^j \delta_{jk} V^k = \sum_{j=1}^{3} V^j V^j$$

Now on to <u>Lorentz transformations</u> = Group of linear transformations of spacetime coordinates which leave the proper length $ds^2 = c^2 dt^2 - d\vec{x}^2$ invariant.

We define a 4-vector x^μ with components $(\underset{x^0}{ct}, \underset{x^i,\ i=1,2,3}{\vec{x}})$

The proper length can be written

$$ds^2 = -\sum_{\mu,\nu=0}^{3} dx^\mu \eta_{\mu\nu} dx^\nu$$

where $\eta_{\mu\nu}$ is the <u>Minkowski tensor</u> $\boxed{\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}_{\mu\nu}}$.

For now we think of $\eta_{\mu\nu}$ as a matrix that does not transform under rotations. It's just a convenient way to write ds^2.

We define $\boxed{x_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\nu}$ with components $(-ct, +\vec{x})$.

The proper length can be written $ds^2 = -\sum_{\mu=0}^{3} dx^\mu dx_\mu$.

We also define $\eta^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}_{\mu\nu}$ and write

$$\boxed{x^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} x_\nu}$$

In this way the Minkowski tensor is used to <u>raise and lower indices</u>.

For consistency, $x^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} x_\nu = \sum_{\nu\alpha=0}^{3} \eta^{\mu\nu} \eta_{\nu\alpha} x^\alpha$,

so $\boxed{\sum_{\nu=0}^{3} \eta^{\mu\nu} \eta_{\nu\alpha} = \underbrace{\delta^\mu{}_\alpha}_{=\begin{cases}1 \text{ if } \mu=\alpha \\ 0 \text{ if } \mu\neq\alpha\end{cases}}}$ as can be checked explicitly.

Under Lorentz transformations, 4-vectors transform as

$\boxed{x^\mu \rightarrow \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu}$ — contravariant 4-vector (has an upper index.)

Then $x_\mu = \sum_\nu \eta_{\mu\nu} x^\nu \rightarrow \sum_{\nu,\alpha} \eta_{\mu\nu} \Lambda^\nu{}_\alpha x^\alpha$

$$= \sum_{\nu,\alpha,\beta} \underbrace{\eta_{\mu\nu} \Lambda^\nu{}_\alpha \eta^{\alpha\beta}}_{\text{sometimes written } \Lambda_\mu{}^\beta \text{ after summing over } \nu,\alpha.} x_\beta$$

So, x_μ transforms differently than x^μ.
x_μ is called a covariant 4-vector (has a lower index)

Lorentz transformation of ds^2:

$$ds^2 = \sum_{\mu\nu} dx^\mu \eta_{\mu\nu} dx^\nu \rightarrow \sum_{\mu\nu\alpha\beta} (\Lambda^\mu{}_\alpha dx^\alpha) \eta_{\mu\nu} (\Lambda^\nu{}_\beta dx^\beta)$$

$$= \sum_{\mu\nu\alpha\beta} dx^\alpha (\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta dx^\beta$$

$$= \sum_{\alpha\beta} dx^\alpha \eta_{\alpha\beta} dx^\beta \quad \text{if} \quad \boxed{\sum_{\mu\nu} (\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$$

The group of matrices Λ satisfying $\Lambda^T \eta \Lambda = \eta$ is the Lorentz group $O(3,1)$. The numbers $(3,1)$ refer to the number of $+1$'s and -1's in η, respectively.

Note that $\sum_{\mu\nu\alpha} \left(\eta_{\mu\nu} \Lambda^\nu{}_\alpha \eta^{\alpha\beta} \right) \Lambda^\mu{}_\lambda$

$$= \sum_{\mu\nu\alpha} \left(\Lambda^\mu{}_\lambda \eta_{\mu\nu} \Lambda^\nu{}_\alpha \right) \eta^{\alpha\beta}$$

$$= \sum_\alpha \eta_{\lambda\alpha} \eta^{\alpha\beta} \quad = \delta_\lambda{}^\beta .$$

In other words, $\boxed{\sum_{\nu\alpha} \eta_{\mu\nu} \Lambda^\nu{}_\alpha \eta^{\alpha\beta} = (\Lambda^{-1})^\beta{}_\mu}$

So we can write the transformation of a covariant 4-vector as

$$\boxed{x_\mu \to \sum_\beta (\Lambda^{-1})^\beta{}_\mu x_\beta}$$

A Lorentz tensor can have upper and lower indices, and transforms as

$$T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} \to \sum_{\{\alpha_i, \beta_i\}} \Lambda^{\mu_1}{}_{\alpha_1} \cdots \Lambda^{\mu_n}{}_{\alpha_n} (\Lambda^{-1})^{\beta_1}{}_{\nu_1} \cdots (\Lambda^{-1})^{\beta_m}{}_{\nu_m} T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_m}$$

— Defines an (n,m) tensor.

<u>Exercise</u>: Show that $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are tensors, but as such are invariant under Lorentz transformations.

We now classify fields by their Lorentz transformation laws.

Scalar Fields: $\phi(x) \to \phi(\Lambda^{-1}x)$

Vector Fields: $A^\mu(x) \to \sum_\nu \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$

$$A_\mu(x) \to \sum_\alpha (\Lambda^{-1})^\alpha{}_\mu A_\alpha(\Lambda^{-1}x)$$

Tensor Fields: $T^{\mu_1 \cdots \mu_n}(x) \to \sum_{\{\alpha_i\}} \Lambda^{\mu_1}{}_{\alpha_1} \cdots \Lambda^{\mu_n}{}_{\alpha_n} T^{\alpha_1 \cdots \alpha_n}(\Lambda^{-1}x)$

etc.

Any product of 4-vectors $\sum_\mu A_\mu B^\mu$ is Lorentz invariant:

$$\sum_\mu A_\mu B^\mu \to \sum_{\alpha\beta\mu} \left((\Lambda^{-1})^\alpha{}_\mu A_\alpha\right)\left(\Lambda^\mu{}_\beta B^\beta\right)$$

$$= \sum_{\alpha\mu\beta} A_\alpha (\Lambda^{-1})^\alpha{}_\mu \Lambda^\mu{}_\beta B^\beta$$

$$= \sum_{\alpha\beta} A_\alpha \delta^\alpha{}_\beta B^\beta = \sum_\alpha A_\alpha B^\alpha \qquad \blacksquare$$

More generally, contracting an upper index with a lower index this way, a tensor transforms as though those indices were not there.

To simplify notation, we introduce the Einstein summation convention: $A_\mu B^\mu \equiv \sum_{\mu=0}^{3} A_\mu B^\mu$.

$$T^{\alpha\beta\gamma}T_{\beta\gamma\delta} \equiv \sum_{\beta\gamma=0}^{3} T^{\alpha\beta\gamma}T_{\beta\gamma\delta}$$

Repeated indices are always summed over, i.e. contracted.

Transformation of derivatives:

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial x^\mu}\phi(\Lambda^{-1}x)$$

$$= \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\nu}\phi(\tilde{x}) \text{ , where } \tilde{x}^\nu = (\Lambda^{-1})^\nu{}_\mu x^\mu$$

$$= (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial \tilde{x}^\nu}\phi(\tilde{x})\Big|_{\tilde{x}=\Lambda^{-1}x}$$

This is just how a covariant 4-vector transforms, i.e. $\frac{\partial}{\partial x^\mu}$ can be thought of as transforming like a 4-vector with a lower index.

This motivates some new notation: $\boxed{\partial_\mu \equiv \frac{\partial}{\partial x^\mu}}$

$$\boxed{\partial^\mu \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\nu}}$$

So, $(\partial_\mu\phi)(\partial^\mu\phi) = \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}\eta^{\mu\nu}$

— transforms as a Lorentz scalar

Still more notation: $\boxed{(\partial_\mu\phi)^2 \equiv (\partial_\mu\phi)(\partial^\mu\phi)}$

— It is the only combination of two factors of $\partial_\mu\phi$ that transforms nicely under Lorentz transformations.

Physical interpretation of Lorentz transformations:

Consider rotations: For a rotation about the x^3-axis by an angle θ we would write

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

For small angles this becomes, $\begin{pmatrix} 1 & & & \\ & 1 & -\theta & \\ & \theta & 1 & \\ & & & 1 \end{pmatrix} + \mathcal{O}(\theta^2)$.

We can write the infinitessimal transformation matrix as $\delta^\mu{}_\nu + \omega^\mu{}_\nu$, where $\omega^\mu{}_\nu$ is the antisymmetric matrix $\begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}$.

For a general infinitessimal rotation by θ about the $\hat{\theta}$ axis we would have for the spatial components $\omega^i{}_j$, $\omega^{ij} = +\omega^i{}_j = -\omega^j{}_i = -\omega^{ji} = -\sum_k \varepsilon^{ijk}\theta\hat{\theta}^k$.

For our rotation about x^3 we have $\omega^{12} = -\theta$.

For a boost in the x^1-direction by velocity v,

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh w & \sinh w & & \\ \sinh w & \cosh w & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where the boost parameter w satisfies

$$\cosh w = 1 \Big/ \sqrt{1 - v^2/c^2} \equiv \gamma$$

$$\tanh w = v/c$$

For small $\frac{v}{c}$, the transformation matrix becomes,

$$\begin{pmatrix} 1 & v/c & & \\ v/c & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \Theta(v/c)^2$$

If we again write this as $\delta^\mu{}_\nu + w^\mu{}_\nu$, then in this example $w^0{}_1 = w^1{}_0 = -w^{10} = +w^{01} = v/c$.

A general infinitessimal Lorentz transformation is specified by the <u>antisymmetric matrix $w^{\mu\nu}$</u>.

Count # parameters in $w^{\mu\nu}$: $\frac{4 \cdot 3}{2} = 6$

$= 3$ rotations $+ 3$ boosts ✓

We can understand the antisymmetry of $\omega_{\mu\nu}$ from the defining relation for Lorentz transformations:

$$\Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \omega^\nu{}_\mu \quad , \quad \Lambda^\mu{}_\nu \eta_{\mu\beta} \Lambda^\beta{}_\alpha = \eta_{\nu\alpha}$$

$$(\delta^\mu{}_\nu + \omega^\mu{}_\nu)\eta_{\mu\beta}(\delta^\beta{}_\alpha + \omega^\beta{}_\alpha)$$

$$= \eta_{\nu\alpha} + \omega_{\alpha\nu} + \omega_{\nu\alpha} + \underbrace{\omega_{\beta\nu}\omega^\beta{}_\alpha}_{\mathcal{O}(\omega^2)}$$

(Note that we have been raising and lowering indices with $\eta_{\mu\beta}$.)

So to linear order in ω, $\boxed{\omega_{\alpha\nu} = -\omega_{\nu\alpha}}$, as promised.

Including translations, $x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{6\text{ parameters}} x^\nu + \underbrace{a^\mu}_{4\text{ parameters}}$

$\rightarrow$ 10-parameter family of Poincaré transformations

Note that there are the same number of parameters in the Poincaré transformations as in the Galilean transformations.

For now let $c=1$ and write $d\tau = ds$ for the proper time = proper length.

Consider the rest frame of a clock, with time interval between clicks $(\Delta\tau)^2 = (\Delta t)^2$.

In a boosted frame, moving w/ speed v in the x-direction with respect to the clock's frame,

$$
\begin{aligned}
(\Delta\tau)^2 &= (\Delta t')^2 - (\Delta x')^2 \\
&= (\Delta t')^2 - (v\Delta t')^2 \\
&= (\Delta t')^2(1-v^2)
\end{aligned}
$$

$\rightarrow \boxed{\Delta t' = \gamma \Delta t}$ Time Dilation