


## How to Make a Black Hole

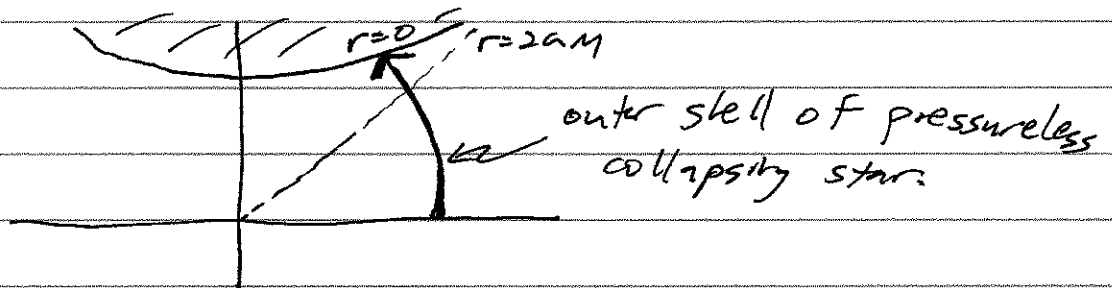
### Collapse of Pressureless Dust

  $r > R: ds^2 = \frac{32 (GM)^3}{r} e^{-r/2GM} (-dT^2 + dX^2) + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$

Each dust particle on the surface (shell) follows a geodesic toward the center:

$$\begin{cases} r = \frac{R}{2} (1 + \cos \eta) \\ T = \frac{R}{2} \left( \frac{R}{2GM} \right)^{1/2} (\eta + \sin \eta) \end{cases}$$

Kruskal coordinates:  $X^2 - T^2 = \left( \frac{r}{2GM} - 1 \right) e^{r/2GM}$



Collapse happens in finite proper time  $\tau$ , but light signals are delayed so outside observers do not see the shell cross the Schwarzschild radius.

## Interior Solutions for Stars

Special Relativity:  $T^{\alpha\beta} = \sum_n p_n(t) \frac{dx_n^\alpha}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$

$T^{00} = \sum_n E_n(t) \delta^3(\vec{x} - \vec{x}_n(t))$  Energy density

$T^{0i} = \sum_n E_n(t) \frac{dx_n^i}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$  Energy Flux in direction  $i$

$T^{ji} = \sum_n p_n^j(t) \frac{dx_n^i}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$  Flux of  $j$ -component of momentum in direction  $i$  (stress)

Fluids:  $N \rightarrow \infty$  limit

Perfect fluid: At each point there is a velocity  $\vec{v}$  such that an observer moving w/ velocity  $\vec{v}$  observes the fluid as isotropic (the same in every direction).

Local rest frame:  $T^{00} = \rho$  Energy density

$T^{0i} = 0$

$T^{ji} = p \delta^{ji}$  ← rotation-invariant

↑ pressure

In local rest frame,  $U^\alpha = \frac{dx^\alpha}{d\tau} = (1, 0, 0, 0)$  4-velocity of fluid element

$$T^{\alpha\beta} = p \eta^{\alpha\beta} + (p + \rho) U^\alpha U^\beta$$

- tensor eqn. → valid in all frames

check:  $T^{00} = -p + (p + \rho) = \rho$

$T^{0i} = 0$

$T^{ji} = p \delta^{ji}$  ✓

$$p_n^M = E_n \frac{dx_n^M}{dt} \rightarrow T^{\alpha\beta} = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

Local rest frame:  $T^{00} = \sum_n E_n \delta^3(\vec{x} - \vec{x}_n(t))$

$$p = \frac{1}{3}(T^{11} + T^{22} + T^{33}) = \frac{1}{3} \sum_n \frac{\vec{p}_n \cdot \vec{p}_n}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

Massive particles:  $E > \frac{|\vec{p}|^2}{E} \Rightarrow \rho \geq 3p \geq 0$

Massless particles (photons):  $E = \frac{|\vec{p}|^2}{E} \Rightarrow \rho = 3p$

General Relativity:  $T^{\alpha\beta} = p g^{\alpha\beta} + (p + \rho) U^\alpha U^\beta$

$$U^\alpha = \frac{dx^\alpha}{d\tau}, \quad g_{\alpha\beta} U^\alpha U^\beta = -1$$

Stars: Static, spherically symmetric solution to Einstein's equations

Look for solutions of the form

$$ds^2 = -e^{2\phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

Fluid elements are at rest  $\Rightarrow U^\alpha = (e^{-\phi(r)}, 0, 0, 0)$

$$g_{\alpha\beta} U^\alpha U^\beta = -1 \quad \checkmark$$

$$\begin{cases} T^{M\nu} = g^{M\nu} p(r) + (p(r) + \rho(r)) U^M U^\nu \\ R^{M\nu} - \frac{1}{2} g^{M\nu} R = -8\pi G T^{M\nu} \end{cases}$$

Define  $m(r)$  by  $e^{2\lambda(r)} \equiv \left(1 - \frac{2Gm(r)}{r}\right)^{-1}$

- like Schwarzschild

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Einstein equations:

(1)  $\frac{dm}{dr} = 4\pi r^2 \rho$

(2)  $\frac{dp}{dr} = -\frac{G\rho m}{r^2} \left(1 + \frac{p}{\rho}\right) \left(1 + \frac{4\pi\rho r^3}{m}\right) \left(1 - \frac{2Gm}{r}\right)^{-1}$

- Tolman-Oppenheimer-Volkov eqn. of hydrostatic equilibrium

(3)  $\frac{d\phi}{dr} = \frac{Gm}{r^2} \left(1 + \frac{4\pi\rho r^3}{m}\right) \left(1 - \frac{2Gm}{r}\right)^{-1}$

Solutions:

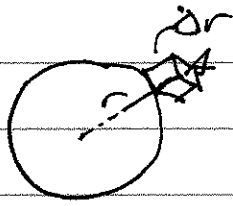
(1)  $\Rightarrow m(r) = 4\pi \int_0^r \rho(r') r'^2 dr' + \text{const.}$

Good behavior at  $r=0$ :  $e^{2\lambda(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1} \Rightarrow m(0) = 0$

Assume  $2Gm(r) \ll r$  for all  $r$ .

(2): To understand the meaning of the Tolman-Oppenheimer-Volkov equation, consider the Newtonian limit,  $\begin{cases} p \ll \rho \\ Gm(r) \ll r \end{cases}$

$\Rightarrow \frac{dp}{dr} \approx -\frac{G\rho m(r)}{r^2}$  — Equation of hydrostatic equilibrium



Gravitational force on infinitesimal box:

$$F_g = \frac{G(A \Delta r \rho(r)) m(r)}{r^2}$$

$$= -(\rho(r + \Delta r)A - \rho(r)A)$$

Divide by  $\Delta r$ :  $-\frac{d\rho}{dr} = \frac{G\rho m(r)}{r^2}$  hydrostatic equilibrium

(3) To understand the meaning of eq. (3), consider the Newtonian limit again.

$$\frac{d\phi}{dr} \approx \frac{Gm(r)}{r^2} \rightarrow \phi(r) = G \int_0^r \frac{m(r')}{r'^2} dr'$$

= Newtonian potential

$$e^{2\phi(r)} \approx 1 + 2\phi(r)$$

Not necessarily in the Newtonian limit!

At the edge of the star  $p = 0$

$$r > R : p = \rho = 0 \rightarrow \frac{d\phi}{dr} = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1}$$

$$\rightarrow e^{2\phi} = 1 - \frac{2GM}{r} \quad \text{Schwarzschild solution}$$

To model a star: (1) Assume an equation of state  $p(\rho)$ .

(2) Assume a central value for  $\rho$ , i.e.  $\rho(0) = \rho_c \rightarrow p(0) = p(\rho_c)$

(3) Integrate Einstein eqs<sup>1+2</sup> to larger  $r$ , s.t.  $m(0) = 0$ .

(4) End integration at  $r$  s.t.  $p(r) = 0 \rightarrow r = R$ .

(5) Integrate Einstein eqn. 3 to find  $\phi(r)$ .

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Example: Incompressible Fluid

$\rho = \text{constant}$ , indep. of  $P$

$$\rho(r) = \begin{cases} \rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

$$m(r) = \frac{4}{3} \pi \rho_0 r^3$$

$$\frac{dP}{dr} = -G \frac{4}{3} \pi \rho_0^2 r \left(1 + \frac{P}{\rho_0}\right) \left(1 + \frac{3P}{\rho_0}\right) \left(1 - \frac{8}{3} \pi \rho_0 G r^2\right)^{-1}$$

Integrate:

$$P(r) = \rho_0 \frac{\left[ \left(1 - \frac{2GM}{R}\right)^{1/2} - \left(1 - \frac{2GM r^2}{R^3}\right)^{1/2} \right]}{\left(1 - \frac{2GM r^2}{R^3}\right)^{1/2} - 3 \left(1 - \frac{2GM}{R}\right)^{1/2}}$$

$P(R) = 0 \rightarrow$  defines  $R$  (edge of star)

$r=0$ : Denominator of  $P(r)$  would vanish if

$$1 - 3 \left(1 - \frac{2GM}{R}\right)^{1/2} = 0$$

$$\rightarrow \frac{GM}{R} = \frac{4}{9}$$

If  $\frac{GM}{R} > \frac{4}{9}$  then  $P(r) \rightarrow \infty$  for some  $r > 0$

—unphysical

$$\rightarrow \boxed{GM < \frac{4R}{9}}$$

Bound on mass of star

In fact, the bound  $GM < \frac{4R}{9}$  follows more generally as long as  $\rho(r) > 0$  and  $\frac{d\rho}{dr} < 0$ .

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White Dwarfs - old star burns its fuel, cools and contracts.

Low temperature  $T \rightarrow$  electrons freeze into lowest available energy levels.

Number of electrons/unit volume:

$$n = 2 \cdot \frac{4\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^2 dk$$

$\uparrow$  spins       $\uparrow$  from density of states       $\leftarrow$  Fermi momentum  $k_F$

$$\Rightarrow n = \frac{k_F^3}{3\pi^2\hbar^3}$$

Mass density mainly from nucleons:

$$\rho = n m_N \mu$$

$\uparrow$  avg. nucleon mass       $\leftarrow$  # nucleons/electron

Iron:  $\mu = \frac{56}{26}$

$$\Rightarrow k_F = \hbar \left( \frac{3\pi^2 \rho}{m_N \mu} \right)^{1/3}$$

Electron pressure:

$$p = \frac{8\pi}{3(2\pi\hbar)^3} \int_0^{k_F} \frac{k^2}{\sqrt{k^2 + m_e^2 c^2}} k^2 dk$$

$\uparrow \frac{p^2}{E}$  (from energy-momentum tensor  $T_{ij}$ )

If  $k_F \gg m_e c$ :

$$p \approx \frac{8\pi k_F^4}{12(2\pi\hbar)^3} = \frac{\hbar}{12\pi^2} \left( \frac{3\pi^2}{m_N \mu} \right)^{4/3} \rho^{4/3}$$

$$\equiv K \rho^{4/3} \quad \text{Equation of state}$$

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- 1) Integrating the equation for hydrostatic equilibrium with this equation of state gives  $p(r)$ .
- 2) pressure vanishes  $\rightarrow$  radius of star  $R$ .
- 3)  $M = \int_0^R 4\pi r'^2 \rho(r') dr'$

$$\Rightarrow M = \frac{1}{2} (3\pi)^{4/2} (2.01824) \left( \frac{\hbar^{3/2} c^{3/2}}{G^{3/2} m_N^2 \mu^2} \right)$$

$$= \boxed{5.87 \mu^{-2} M_{\odot} \equiv M_C}$$

For smaller  $k_F$ ,  $M < M_C$ . Hence, white dwarf stars have mass  $< M_C \equiv$  Chandrasekhar limit.

In fact, for  $k_F \approx 5 m_e c$ , it is energetically favorable for electrons to be captured by nuclei, converting protons to neutrons.  
 $\Rightarrow M < 1.2 M_{\odot}$ , close to Chandrasekhar limit of  $\mu = \frac{56}{26}$