Coordinate Singularities

Wald 6.4

In the Schwarzschild spacetime curvature invariants like $R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}$ behave regularly around $r = 2GM$, but the metric depends on $\left(1 - \frac{2GM}{r}\right)^{1/2}$ which behaves singularly.

This is an example of a coordinate singularity, which can be eliminated by a change of coordinates. A simple analogy is provided by the metric

$$ds^2 = -\frac{1}{t^4} dt^2 + dx^2 , \quad \begin{matrix} -\infty < x < \infty \\ 0 < t < \infty \end{matrix}$$

A change of coordinates $t \to t' = 1/t$ removes the singularity at $t = 0$:

$$ds^2 = (t')^4 \left(\frac{-dt'}{(t')^2}\right)^2 + dx^2$$

$$= -(dt')^2 + dx^2 \qquad \text{Minkowski space.}$$

The region covered by the original coordinates $0 < t < \infty$ is the upper-half plane in Minkowski space, $0 < t' < \infty$.

The spacetime described by the original metric is <u>geodesically complete</u> as $t \to 0$, meaning all geodesics approaching $t = 0$ extend to arbitrarily values of their affine parameter τ.

However, geodesics may reach $t=\infty$ for finite values of their affine parameter, so the coordinates (x,t) do not describe a geodesically complete spacetime.

On the other hand, the geometry described by (x,t') may be made geodesically complete by extending the coordinate range from $0<t'<\infty$ to $-\infty<t'<\infty$.

Another example, similar in some respects to the Schwarzschild case, is the Rindler Spacetime,

$$ds^2 = -x^2 dt^2 + dx^2 , \begin{cases} -\infty < t < \infty \\ 0 < x < \infty \end{cases}$$

The metric appears singular at $x=0$. Geodesics terminate with finite affine parameter at $x=0$, but the curvature is regular as $x \to 0$. Indeed, $R_{\mu\nu\lambda\sigma}=0$ everywhere in the spacetime.

Null geodesics: $-x^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = 0$

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

$$t = \pm \ln x + \text{const.}$$

$+$ = "outgoing"
$-$ = "ingoing"

Define coordinates $u = t - \ln x$, $-\infty < u < \infty$
$v = t + \ln x$, $-\infty < v < \infty$

$v - u = 2 \ln x$

$x^2 = e^{v-u}$

$v + u = 2t$

$t = \frac{v+u}{2}$

$dx = \frac{1}{2} e^{\frac{v-u}{2}} dv - \frac{1}{2} e^{\frac{v-u}{2}} du$

$dt = \frac{1}{2} dv + \frac{1}{2} du$

$$ds^2 = -x^2 dt^2 + dx^2$$

$$= -\frac{e^{v-u}}{4}\left[(dv^2 + du^2 + 2 du dv)\right] + \frac{e^{v-u}}{4}\left[dv^2 + du^2 - 2du dv\right]$$

$$\boxed{ds^2 = -e^{v-u} du dv}, \quad \begin{cases} -\infty < u < \infty \\ -\infty < v < \infty \end{cases}$$

Metric indep. of t $\rightarrow$ Killing vector $\xi^\mu = \delta^\mu{}_t$.

Constant of the motion: $\tilde{E} = -g_{\mu t} \frac{dx^\mu}{d\tau}$

$$= x^2 \frac{dt}{d\tau}$$

$$= e^{v-u}\left(\frac{1}{2}\frac{dv}{d\tau} + \frac{1}{2}\frac{du}{d\tau}\right)$$

u = constant : "outgoing" trajectory

$$\tau_{out} = \frac{1}{2\tilde{E}} \int e^{v-u} dv$$

$$= \text{Const.} + \underbrace{\left(\frac{e^{-u}}{2\tilde{E}}\right)}_{\text{const.}} e^{v}$$

v = const: "ingoing" geodesic

$$\tau_{in} = \frac{1}{2\tilde{E}} \int e^{v-u}\, du = \text{Const.} - \left(\frac{e^{v}}{2\tilde{E}}\right) e^{-u}$$

const.

τ_{in} and τ_{out} motivate a new choice of coordinates:

$$U = -e^{-u}, \quad V = e^{v} \quad , \begin{cases} -\infty < U < 0 \\ 0 < V < \infty \end{cases}$$

$$\Rightarrow \boxed{ds^2 = -dU\, dV}$$ "Null geodesic coordinates"

By extending the coordinate range to $\begin{cases} -\infty < U < \infty, \\ -\infty < V < \infty \end{cases}$

the spacetime becomes geodesically complete.

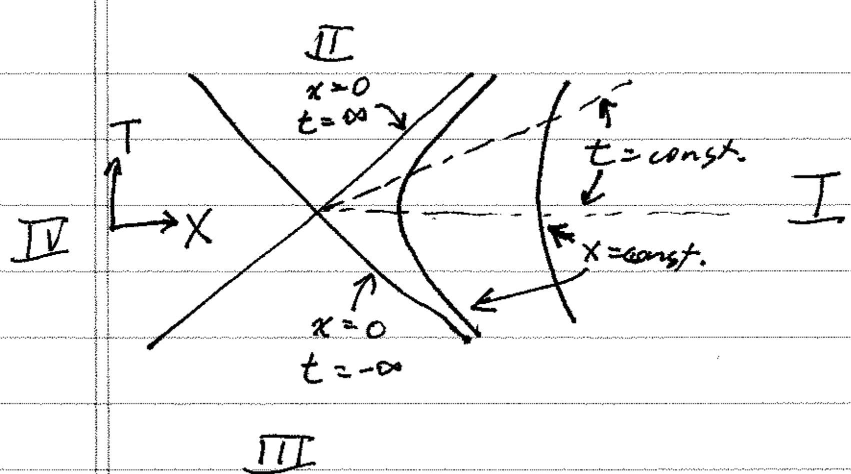
To show that the geometry extended in this way is just Minkowski space, define

$$T = \frac{U+V}{2}, \quad X = \frac{V-U}{2}$$

$$\Rightarrow \boxed{ds^2 = -dT^2 + dX^2} \quad , \begin{cases} -\infty < T < \infty \\ -\infty < X < \infty \end{cases}$$

In terms of the original coordinates,

$$\boxed{\begin{aligned} x &= (X^2 - T^2)^{1/2} \\ t &= \tanh^{-1}(T/X) \end{aligned}}$$



Rindler spacetime is the region I $(X > |T|)$ of Minkowski spacetime.

Consider a (non geodesic) trajectory $x = \text{const.}$ in the original coordinates.

The proper acceleration is $\boxed{a^\mu = \frac{D}{D\tau}\left(\frac{dx^\mu}{d\tau}\right) = U^\nu D_\nu U^\mu}$
where $U^\mu = \frac{dx^\mu}{d\tau}$,
and $U^\mu U^\nu g_{\mu\nu} = -1 \Longrightarrow U^\mu = \left(\frac{1}{x}, 0\right)$.

The nonvanishing Christoffel symbols are

$$\Gamma^t_{xt} = \Gamma^t_{tx} = -1/x$$

$$\Gamma^x_{tt} = x$$

$$a^\mu = U^\nu\left(\partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda\right)$$
$$= (U^t)^2 \Gamma^\mu_{tt} = \frac{1}{x^2}\Gamma^\mu_{tt}$$

$\boxed{a^x = \frac{1}{x}}$, $a^t = 0$

$\longrightarrow$ The Rindler coordinates (x, t) describe Minkowski space in an accelerated coordinate system.

Kruskal Coordinates

Consider the r, t part of the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

Null geodesics satisfy

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 0$$

$$\Longrightarrow \left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2GM}{r}\right)^{-2}$$

Solutions: $\boxed{t = \pm r_* + \text{constant},}$

where $\boxed{r_* = r + 2GM \ln\left(r/2GM - 1\right)}$

is the "Regge-Wheeler tortoise coordinate."

Note that $\frac{dr_*}{dr} = \left(1 - \frac{2GM}{r}\right)^{-1}$.

Define null coordinates u, v:

$$u = t - r_*$$
$$v = t + r_*$$

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv \quad \text{, where } r = r(u,v)$$

from $r_* = \frac{v-u}{2}$

$$\rightarrow r + 2GM \ln\left(\frac{r}{2GM} - 1\right) = \frac{v-u}{2}$$

$$\frac{r}{2GM} - 1 = e^{(v-u)/4GM}\, e^{-r/2GM}$$

$$= \frac{r}{2GM}\left(1 - \frac{2GM}{r}\right)$$

$$\rightarrow ds^2 = -\underbrace{\frac{2GM\, e^{-r/2GM}}{r}}_{\text{Nonsingular as } r\to 2GM,\ (u\to\infty \text{ or } v\to-\infty)} e^{(v-u)/4GM}\, du\, dv$$

Define new coordinates $U = -e^{-u/4GM}$
$V = e^{v/4GM}$

$$\Rightarrow ds^2 = -\frac{32(GM)^3 e^{-r/2GM}}{r}\, dU\, dV$$

No singularity at $r=2GM$ ($U=0$ or $V=0$)

Change coordinates to $T = \frac{U+V}{2}$, $X = \frac{V-U}{2}$, and restore the angular coordinates:

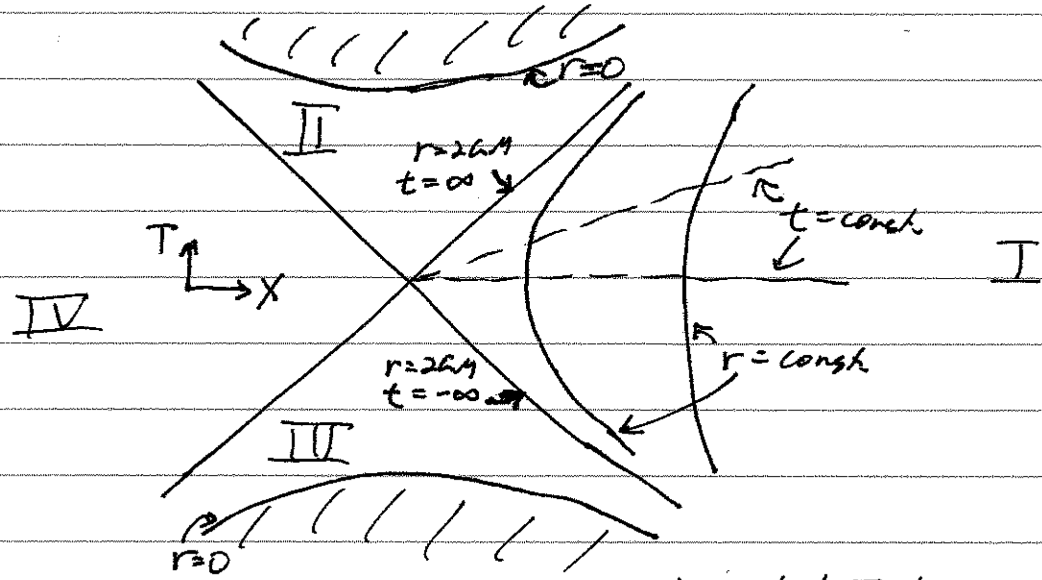
$$\boxed{ds^2 = \frac{32(GM)^3 e^{-r(u,v)/2GM}}{r(u,v)}\left(-dT^2 + dX^2\right) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

— Schwarzschild metric in Kruskal coordinates

$$\boxed{\begin{gathered}\left(\frac{r}{2GM}-1\right) e^{r/2GM} = X^2 - T^2 \\ \frac{t}{2GM} = \ln\left(\frac{T+X}{X-T}\right) = 2\tanh^{-1}\left(T/X\right)\end{gathered}}$$

The singularity at $r=0$ is a true curvature singularity, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$ as $r\to 0$. The allowed range of coordinates X, T follows from the condition $r>0$

$$\Rightarrow \boxed{X^2 - T^2 > -1}$$



Kruskal Extension of Schwarzschild Spacetime

Region I: $r > 2GM$ of original Schwarzschild spacetime.

Region II: Black Hole – all observers in this region reach the singularity at $r=0$ in finite proper time.

Region III: White Hole – all observers originated at $r=0$ $\left(X = -(T^2-1)^{1/2}\right)$ and leave region III in finite proper time.

Region IV: looks like Region I – asymptotically flat, $r > 2GM$.

$t=0$.

Each circle represents a 2-sphere.