

Killing Vector Fields

In order to describe motion in the Schwarzschild spacetime it will be useful to identify certain constants of the motion. We first digress with a more general discussion.

Consider a space(time) with metric $g_{\mu\nu}$. Under an infinitesimal coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu$,

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x(x')) \\ &= \left(\delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x'^\mu} \right) \left(\delta^\beta_\nu + \frac{\partial \xi^\beta}{\partial x'^\nu} \right) \left(g_{\alpha\beta}(x) + \left(\frac{\partial}{\partial x'^\lambda} g_{\alpha\beta}(x) \right) \xi^\lambda + \mathcal{O}(\xi^2) \right) \\ &= g_{\mu\nu}(x') + g_{\alpha\nu}(x') \frac{\partial \xi^\alpha}{\partial x'^\mu} + g_{\mu\beta}(x') \frac{\partial \xi^\beta}{\partial x'^\nu} + \xi^\lambda \frac{\partial}{\partial x'^\lambda} g_{\mu\nu}(x') + \mathcal{O}(\xi^2) \\ &= g_{\mu\nu}(x') + \left[\frac{\partial}{\partial x'^\mu} (g_{\alpha\nu} \xi^\alpha) - \xi^\alpha \frac{\partial}{\partial x'^\mu} g_{\alpha\nu} \right] + \left[\frac{\partial}{\partial x'^\nu} (g_{\mu\beta} \xi^\beta) - \xi^\beta \frac{\partial}{\partial x'^\nu} g_{\mu\beta} \right] \\ &\quad + \xi^\lambda \frac{\partial}{\partial x'^\lambda} g_{\mu\nu} + \mathcal{O}(\xi^2) \end{aligned}$$

Dropping terms
of $\mathcal{O}(\xi^2)$

$$\begin{aligned} &= g_{\mu\nu} + \frac{\partial \xi_\nu}{\partial x'^\mu} + \frac{\partial \xi_\mu}{\partial x'^\nu} - \xi^\lambda g^{\alpha\lambda} \left(\frac{\partial g_{\alpha\nu}}{\partial x'^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x'^\nu} - \frac{\partial g_{\mu\nu}}{\partial x'^\alpha} \right) \\ &= g_{\mu\nu} + \frac{\partial \xi_\nu}{\partial x'^\mu} + \frac{\partial \xi_\mu}{\partial x'^\nu} - 2 \xi^\lambda \Gamma^\lambda_{\mu\nu} \end{aligned}$$

$$g'_{\mu\nu} = g_{\mu\nu} + \Delta_\mu \xi_\nu + \Delta_\nu \xi_\mu$$

A transformation of the coordinates which leaves the form of the metric invariant is called an isometry.

For example, in $3D$ Euclidean space in Cartesian coordinates x, y, z :

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij} dx^i dx^j,$$

where $g_{ij} = \delta_{ij}$

Translations of the coordinates leave lengths invariant:

$$x^i \rightarrow x'^i = x^i + c^i \quad \text{for constant } c^i$$

— leaves $g_{ij} = \delta_{ij}$.

This corresponds to the translational invariance of the

Euclidean geometry, which "looks the same" when translated. Similarly, rotations of the coordinates leave

$g_{ij} = \delta_{ij}$, corresponding to the rotational invariance of the Euclidean geometry.

In classical mechanics we are familiar with the consequences of such symmetries: translational invariance \rightarrow constant momentum, rotational invariance \rightarrow constant $\&$ momentum.

The existence of isometries in a spacetime has similar consequences for freely falling trajectories.

An infinitesimal coordinate transformation specified by $\xi^\mu(x)$ is an isometry if

$$\left\{ D_\mu \xi_\nu + D_\nu \xi_\mu = 0 \right\} \quad \text{Killing Equations.}$$

A solution to this equation is called a Killing vector field.

Consider a geodesic with tangent vector $U^\mu = \frac{dx^\mu}{d\tau}$,
 where τ is the proper time (for a massive object).

Along a geodesic $\frac{DU^\mu}{d\tau} = \frac{dx^\nu}{d\tau} D_\nu U^\mu = \boxed{U^\nu D_\nu U^\mu = 0}$.

Consider $C \equiv \xi_\mu U^\mu$, where ξ^μ is a Killing vector field.

$$\frac{dC}{d\tau} = \frac{DC}{d\tau} = U^\mu \frac{D\xi_\mu}{d\tau} + \xi_\mu \frac{DU^\mu}{d\tau}$$

$$= U^\mu U^\nu D_\nu \xi_\mu + \xi_\mu U^\nu D_\nu U^\mu$$

$$= (0 \text{ by Killing eqn.}) + (0 \text{ by geodesic eqn.})$$

$$\Rightarrow \boxed{\frac{d}{d\tau} \left(\xi_\mu \frac{dx^\mu}{d\tau} \right) = 0}$$

★ Hence, $\xi_\mu \frac{dx^\mu}{d\tau} = C$ is a constant of the motion.

Suppose a metric is independent of one of the coordinates, say x^0 , so that $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$.

Consider the vector field $\xi^\mu = \delta^\mu_0$.

$$\xi_\alpha = g_{\alpha\mu} \xi^\mu = g_{\alpha 0}$$

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \underbrace{g^{\alpha\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})}_{2\Gamma_{\mu\nu}^\alpha} \underbrace{g_{\alpha 0}}_{\xi_\alpha}$$

Using $g^{\alpha\lambda} g_{\lambda 0} = \delta^\alpha_0$

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - (\partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}) = 0$$

Hence, ξ^μ is a Killing vector field.

Motion in the Schwarzschild Spacetime

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$= -d\tau^2$ (for massive object)

The metric is independent of t and φ , so we can identify two constants of the motion.

$$t: \quad \xi^M = \delta^M_0 \rightarrow \xi_M \frac{dx^M}{d\tau} = g_{Mt} \frac{dx^M}{d\tau} = -\left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} \equiv -\tilde{E}$$

$$\boxed{\tilde{E} = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau}}$$
 is a constant of the motion
= "Energy/unit mass."

$$\varphi: \quad \xi^M = \delta^M_\varphi \rightarrow \xi_M \frac{dx^M}{d\tau} = g_{M\varphi} \frac{dx^M}{d\tau} = r^2 \sin^2\theta \frac{d\varphi}{d\tau} \equiv \tilde{L}$$

$$\boxed{\tilde{L} = r^2 \sin^2\theta \frac{d\varphi}{d\tau}}$$
 is a constant of the motion
= "Angular momentum/unit mass"

Along the trajectory of a test particle,

$$\left(1 - \frac{2GM}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 - r^2 \sin^2\theta \left(\frac{d\varphi}{d\tau}\right)^2 = K$$

where $K=1$ for massive object (particle)
 $K=0$ for massless particle

By the symmetry $\theta \rightarrow \pi - \theta$, if motion begins in the $\theta = \pi/2$ plane, it will remain in that plane. Choose coordinates so that $\theta = \pi/2$.

$$\Rightarrow \left(1 - \frac{2GM}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \overset{1}{\sin^2\theta} \left(\frac{d\varphi}{d\tau}\right)^2 = K$$

$$\frac{\tilde{E}^2}{(1-\frac{2GM}{r})} - \frac{1}{(1-\frac{2GM}{r})} \left(\frac{dr}{dt}\right)^2 - \frac{\tilde{L}^2}{r^2} = K$$

$$\frac{1}{2} \left(\frac{dr}{dt}\right)^2 + \frac{1}{2} \left(1 - \frac{2GM}{r}\right) \left(K + \frac{\tilde{L}^2}{r^2}\right) = \frac{1}{2} \tilde{E}^2$$

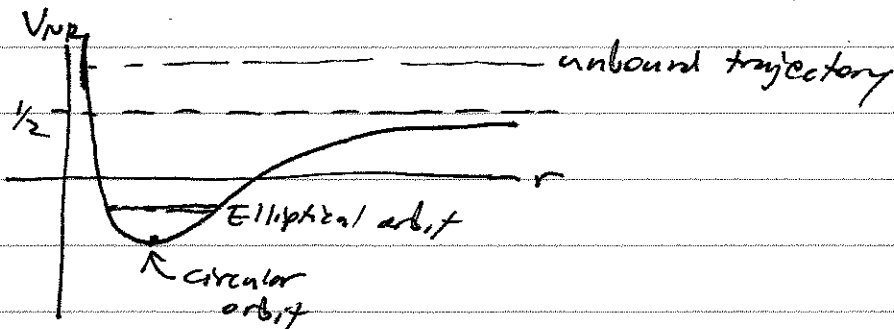
Compare with the Newtonian Limit:

$K=1$, ignore $\frac{1}{r^3}$ term, $\tau=t$

$$\frac{1}{2} \left(\frac{dr}{dt}\right)^2 + \left[\frac{1}{2} - \frac{GM}{r} + \frac{1}{2} \frac{\tilde{L}^2}{r^2} \right] = \frac{1}{2} \tilde{E}^2$$

$V_{NR}(r)$ nonrelativistic effective potential

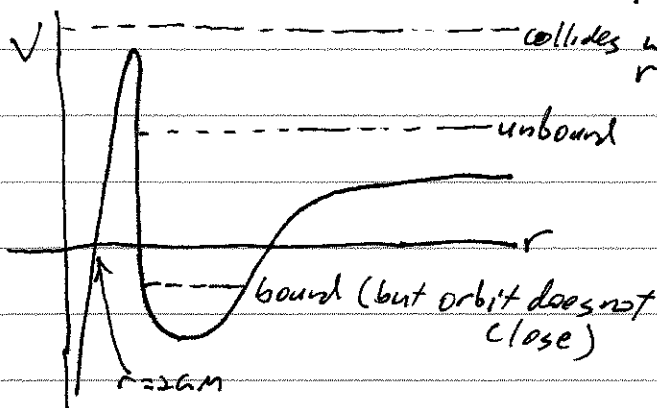
The values of $\tilde{E}^2/2, \tilde{L}^2$ determines the type of trajectory



The $1/r^3$ which we dropped is a correction from G.R.

$$V(r) = \frac{1}{2} K - \frac{KGM}{r} + \frac{\tilde{L}^2}{2r^2} - \frac{GM}{r^3} \tilde{L}^2$$

$K=1$:



$k=0$;

