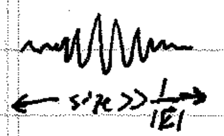
Power Radiated in Gravitational Waves

Consider a plane wave $h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik\cdot x} + \epsilon^*_{\mu\nu} e^{-ik\cdot x}$.
The energy-momentum "tensor" in the wave pulse is given by $t_{\mu\nu} = \frac{1}{8\pi G_N}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \left(R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}\right)\right]$

The expansion of $t_{\mu\nu}$ in $\epsilon_{\mu\nu}$ is messy, but simplifies if we consider the spatial average over distances $\gg \frac{1}{|k|}$, $\langle t_{\mu\nu}\rangle$. Then we can use $\langle e^{\pm 2ik\cdot x}\rangle = 0$, which eliminates a number of terms.



$\langle t_{\mu\nu}\rangle$ = spatial average over pulse.

Using $R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^\lambda_\lambda - \partial_\lambda\partial_\nu h^\lambda{}_\mu - \partial_\lambda\partial_\mu h^\lambda{}_\nu + \partial_\lambda\partial^\lambda h_{\mu\nu}\right)$,

$$R^{(1)} = \partial_\lambda\partial^\lambda h - \partial_\lambda\partial_\rho h^{\lambda\rho}$$

$$\boxed{R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = \frac{1}{2}\Big(\partial_\mu\partial_\nu h - \partial_\lambda\partial_\nu h^\lambda{}_\mu - \partial_\lambda\partial_\mu h^\lambda{}_\nu + \partial_\lambda\partial^\lambda h_{\mu\nu} - \eta_{\mu\nu}\partial_\lambda\partial^\lambda h^\rho{}_\rho + \eta_{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho}\Big)}$$

Comparing with our earlier analysis of plane waves, $R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = 0$, which confirms the linearized approximation of the vacuum Einstein eqns.

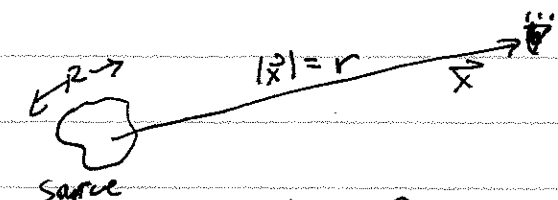
Using our earlier expression for $R_{\mu\nu}^{(2)}$ and $\langle e^{\pm 2ik\cdot x}\rangle = 0$, we obtain (Exercise):

$$\boxed{\langle R_{\mu\nu}^{(2)}\rangle = \frac{k_\mu k_\nu}{2}\left(\epsilon^{\lambda\rho}\epsilon_{\lambda\rho}^{*} - \tfrac{1}{2}|\epsilon^{\lambda}{}_{\lambda}|^2\right)}$$

Using $k^2=0$, $\boxed{\langle R^{(2)}\rangle = 0}$

Hence, $\boxed{\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G_N}\left(\epsilon^{\lambda\rho}\epsilon_{\lambda\rho}^{*} - \tfrac{1}{2}|\epsilon^{\lambda}{}_{\lambda}|^2\right)}$

We have previously found the gravitational radiation field far from a localized oscillating source. Summarizing our earlier results:



$$T_{\mu\nu}(\vec{x},t) = e^{-i\omega t}\,\tilde{T}_{\mu\nu}(\vec{x},\omega) + c.c.$$

$$\bar{h}_{\mu\nu}(\vec{x},t) \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h^{\rho}{}_{\rho}$$
$$= -\frac{\lambda}{4\pi r}\tilde{T}_{\mu\nu}(\vec{k},\omega)\,e^{ik\cdot x}$$

where we had, in our new language, $R_{\mu\nu}^{(1)} - \tfrac{1}{2}\eta_{\mu\nu}R^{(1)} = -\frac{\lambda}{2}T_{\mu\nu}$

$$\Rightarrow \boxed{\lambda = -16\pi G_N}$$

$$\Rightarrow \bar{h}_{\mu\nu}(\vec{x},t) = \frac{4G}{r}e^{-i\omega(t-r)}\tilde{T}_{\mu\nu}(\vec{k},\omega) \quad \text{at } \vec{k} = \omega\hat{x}$$

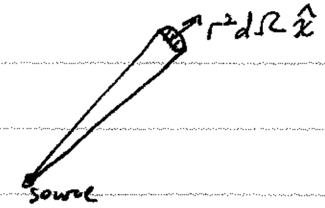
$$\boxed{\epsilon_{\mu\nu}(\vec{x},\omega) = \frac{4G}{r}\left(\tilde{T}_{\mu\nu}(\vec{k},\omega) - \tfrac{1}{2}\eta_{\mu\nu}\tilde{T}(\vec{k},\omega)\right)}$$

$$\epsilon_{\mu\nu}^{*}\epsilon^{\mu\nu} = \frac{16G^2}{r^2}\left(\tilde{T}_{\mu\nu}^{*}\tilde{T}^{\mu\nu} - \frac{1}{2}\tilde{T}^{*}\eta_{\mu\nu}\tilde{T}^{\mu\nu} - \frac{1}{2}\tilde{T}_{\mu\nu}^{*}\eta^{\mu\nu}\tilde{T} + \frac{1}{4}\underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{4}\tilde{T}^{*}\tilde{T}\right)$$

$$= \frac{16G^2}{r^2}\tilde{T}_{\mu\nu}^{*}\tilde{T}^{\mu\nu}$$

$$|\epsilon^{\lambda}{}_{\lambda}|^2 = \frac{16G^2}{r^2}\left|\tilde{T}^{\lambda}{}_{\lambda} - \frac{1}{2}\underbrace{\delta^{\lambda}{}_{\lambda}}_{4}\tilde{T}\right|^2 = \frac{16G^2}{r^2}|\tilde{T}|^2$$

$$\rightarrow \boxed{\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G_N}\cdot\frac{16G_N^2}{r^2}\left(\tilde{T}_{\mu\nu}^{*}\tilde{T}^{\mu\nu} - \frac{1}{2}|\tilde{T}|^2\right)}$$



Away from sources, $\partial_\mu t^{\mu\nu} = 0$

$$\partial_0 t^{00} = -\partial_i t^{i0}$$

$\int d^3x$: $\quad E = \int d^3x\, t^{00}$ $\qquad$ Gauss' law

$$P \equiv -\frac{dE}{dt} = +\int_V d^3x\, \partial_i t^{i0} = +\int_{\partial V} d\Omega\, r^2 \hat{x}^i t_i{}^0$$

$\partial V \leftarrow$ boundary of V

$P =$ radiated power in gravitational radiation from vol. V

$$\frac{dP}{d\Omega} = r^2\hat{x}^i\langle t^{i0}\rangle$$

$$= \frac{r^2(\vec{k}\cdot\hat{x})k^0}{16\pi G_N}\cdot\frac{16G_N^2}{r^2}\left(\tilde{T}_{\mu\nu}^{*}\tilde{T}^{\mu\nu} - \frac{1}{2}|\tilde{T}|^2\right)$$

$$= \frac{G_N\omega^2}{\pi}\left(\tilde{T}_{\mu\nu}^{*}(\vec{k},\omega)\tilde{T}^{\mu\nu}(\vec{k},\omega) - \frac{1}{2}|\tilde{T}^{\lambda}{}_{\lambda}(\vec{k},\omega)|^2\right)$$

Recall from earlier that with the relevant approximations,

$$\tilde{T}_{ij}(\hat{k},\omega) = -\frac{\omega^2}{2} D_{ij}(\omega)$$

where D_{ij} is the quadrupole moment

$$D_{ij}(\omega) \equiv \int d^3x \, x^i x^j \, \tilde{T}_{00}(\hat{x},\omega)$$

Using $\partial_\mu T^{\mu\nu} = 0$ (to leading order),

$$\boxed{K_\mu \tilde{T}^{\mu\nu}(\hat{k},\omega) = 0}$$

$$\rightarrow K_0 \tilde{T}^{0i} + k_j \tilde{T}^{ji} = 0$$

$$\boxed{\tilde{T}^{0i} = -\frac{k_j}{K_0} \tilde{T}^{ji} = \hat{x}_j \tilde{T}^{ji}}$$

$$K_0 \tilde{T}^{00} + \tilde{k}_i \tilde{T}^{i0} = 0$$

$$\boxed{\tilde{T}^{00} = -\frac{\tilde{k}_i}{k_0} \tilde{T}^{i0} = \hat{x}_i \hat{x}_j \tilde{T}^{ij}}$$

Hence, we can express all components of $\tilde{T}_{\mu\nu}$ in terms of $\tilde{T}_{ij}$, and hence in terms of D_{ij}.

$$\boxed{\begin{aligned} \tilde{T}^{0i} &= -\frac{\omega^2}{2} \hat{x}_j D^{ij} \\ \tilde{T}^{00} &= -\frac{\omega^2}{2} \hat{x}_i \hat{x}_j D^{ij} \end{aligned}}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{G_N \omega^2}{\pi} \left(-\frac{\omega^2}{2}\right)^2 \Big(\hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l D^{*ij} D^{kl} - 2 \hat{x}_j \hat{x}_k D^{*ji} D^{ki}$$

$$+ D^{*ij} D_{ij} - \frac{1}{2} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l D^{*ij} D^{kl} + \frac{1}{2} \hat{x}_k \hat{x}_l D^{*kl} D^{ii}$$

$$+ \frac{1}{2} \hat{x}_k \hat{x}_l D^{*ii} D^{kl} - \frac{1}{2} D^{*ii} D^{jj} \Big)$$

$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi}\Big(\frac{1}{2}\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l D^{*ij}D^{kl} - 2\hat{x}_j\hat{x}_k D^{*ji}D^{ki}$$

$$+ D^{*ij}D^{ij} + \frac{1}{2}\hat{x}_k\hat{x}_l D^{*kl}D^{ii} + \frac{1}{2}\hat{x}_k\hat{x}_l D^{*ii}D^{kl}$$

$$- \frac{1}{2}D^{*ii}D^{jj}\Big)$$

Total Power radiated: $P = \int d\Omega \frac{dP}{d\Omega} = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\frac{dP}{d\Omega}$

Use $\int d\varphi\, d(\cos\theta) = 4\pi$

$\int d\varphi\, d(\cos\theta)\, \hat{x}_i\hat{x}_j = \frac{4}{3}\pi\delta_{ij}$

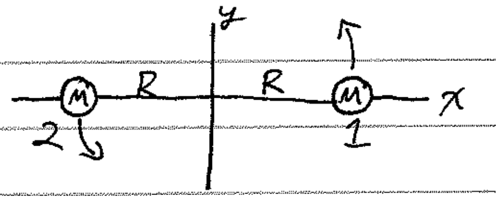
$\int d\varphi\, d(\cos\theta)\, \hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$

$$\Rightarrow \boxed{P = \frac{2G\omega^6}{5}\left(D^*_{ij}(\omega)D_{ij}(\omega) - \frac{1}{3}|D_{ii}(\omega)|^2\right)}$$

This is the total power radiated at one frequency in the quadrupole approximation $\omega R \ll 1$

Binary Star System

Assume a circular Newtonian orbit:



$$\frac{GM^2}{(2R)^2} = \frac{Mv^2}{R}$$

$$v = \frac{2\pi R}{T} \rightarrow \frac{GM}{4} T^2 = (2\pi)^2 R^3$$

$\uparrow$ T = period of orbit.

Let $\Omega = \frac{2\pi}{T} \longrightarrow \frac{GM}{4\Omega^2} = R^3$

Star 1: $x_1 = R\cos\Omega t, \quad y_1 = R\sin\Omega t, \quad z_1 = 0$

Star 2: $x_2 = -R\cos\Omega t, \quad y_2 = -R\sin\Omega t, \quad z_2 = 0$

$$\boxed{\begin{aligned} T^{00} = M\Big(&\delta(z)\,\delta(x - R\cos\Omega t)\,\delta(y - R\sin\Omega t) \\ &+ \delta(z)\,\delta(x + R\cos\Omega t)\,\delta(y + R\sin\Omega t)\Big) \end{aligned}}$$

$$D_{zz} = \int d^3x\, z^2\, T^{00}(\vec{x},t) = 0 = D_{yz} = D_{xz}$$

$$D_{xx} = \int d^3x\, x^2\, T^{00}(\vec{x},t)$$
$$= 2MR^2\cos^2\Omega t = MR^2\left(1 + \tfrac{1}{2}e^{2i\Omega t} + \tfrac{1}{2}e^{-2i\Omega t}\right)$$

$$\rightarrow \boxed{D_{xx}(2\Omega) = \tfrac{1}{2}MR^2}$$

$$D_{yy} = 2MR^2\sin^2\Omega t = MR^2\left(1 - \tfrac{1}{2}e^{2i\Omega t} - \tfrac{1}{2}e^{-2i\Omega t}\right)$$

$$\rightarrow \boxed{D_{yy}(2\Omega) = -\tfrac{1}{2}MR^2}$$

$$D_{xy} = \int d^3x\, xy\, T^{00} = 2MR^2\sin\Omega t\cos\Omega t = MR^2\sin 2\Omega t$$
$$= -i\frac{MR^2}{2}\left(e^{i\Omega t} - e^{-i\Omega t}\right)$$

$$\rightarrow \boxed{D_{xy}(2\Omega) = -i\frac{MR^2}{2}}$$

$$D^*_{ij} D^{ij} = \left(\frac{MR^2}{2}\right)^2 (1+1+1+1) = M^2 R^4$$

$$D^i_i = D_{xx} + D_{yy} = 0$$

Power $\mathcal{P} = \frac{2G}{5}(2\Omega)^6 M^2 R^4 = \frac{128}{5} G M^2 R^4 \Omega^6$

$$\Omega^2 = \frac{GM}{4R^3} \Rightarrow \boxed{\mathcal{P} = \frac{2}{5} \frac{G^4 M^5}{R^5}}$$

Power radiated in gravitational radiation

$E \propto -T^{-2/3}$ for Newtonian orbits

$$\frac{1}{T}\frac{dT}{dt} = -\frac{3}{2}\frac{1}{E}\frac{dE}{dt} = \frac{3}{2}\frac{1}{E}\mathcal{P}$$

1974 Hulse - Taylor Binary Pulsar

-Supernova remnant PSR 1913+16

Change in orbital period measured, Mass and size of orbit known

→ Agrees with prediction of change in period due to energy loss to gravitational radiation!

<u>Note</u>: When comparing with astronomical data, the semimajor axis and other geometric quantities are typically quoted with respect to one of the objects (called the "principal object"). For example, in our example the semimajor axis would have length $2R$.