Weinberg 7.1

Einstein's Field Equations

In the absence of gravitation the energy-momentum tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$.
By the principle of general covariance, we expect the conservation law to be covariant, so that the energy-momentum tensor is instead covariantly conserved,

$$\boxed{D_\mu T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\mu} = 0}$$

In the nonrelativistic, weak field limit, T_{00} is the mass density ρ: $T_{00} \approx \rho$.
We have seen that in this limit $g_{00} \approx -(1+2\phi)$, where ϕ is the Newtonian gravitational potential.

The Poisson eqn. for ϕ is $\nabla^2 \phi = 4\pi G_N \rho$, where $G_N = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$ is Newton's constant.

Combining the above, $\nabla^2 g_{00} \approx -8\pi G_N T_{00}$

It is natural to guess that the covariant form of this equation will take the form

$$\boxed{G_{\mu\nu} = -8\pi G_N T_{\mu\nu}}$$

for some tensor $G_{\mu\nu}$ such that $G_{00} \approx \nabla^2 g_{00}$ in the nonrelativistic limit.

We look for a tensor $G_{\mu\nu}$ such that:

(1) $G_{\mu\nu}$ consists of terms with 2 derivatives of $g_{\mu\nu}$.

(2) $G_{\mu\nu}$ is symmetric in $\mu \leftrightarrow \nu$ (because $T_{\mu\nu}$ is).

(3) $D_\mu G^\mu{}_\nu = 0$ (because $D_\mu T^\mu{}_\nu = 0$).

(4) $G_{00} \approx \nabla^2 g_{00}$ in the nonrelativistic weak-field limit.

The Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$ is the only tensor that can be formed from the metric and its 1st or 2nd derivatives. Hence, the most general tensor satisfying (1) and (2) is made of contractions of $R_{\mu\nu\lambda\sigma}$:

$$G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R$$

for some constants c_1 and c_2.

Recall the Bianchi identity, $D_\mu R^\mu{}_\nu = \frac{1}{2} D_\nu R$.

By condition (3), recall $D^\mu g_{\mu\nu} = 0$

$$0 = D_\mu G^\mu{}_\nu = c_1 D_\mu R^\mu{}_\nu + c_2 g_{\mu\nu} D^\mu R$$

$$= \left(\frac{c_1}{2} + c_2\right) D_\nu R$$

Hence, either $c_2 = -c_1/2$ or $D_\nu R = 0$ everywhere.

But $G^\mu{}_\mu = (c_1 + 4c_2) R = -8\pi G T^\mu{}_\mu$

and $D_\nu T^\mu{}_\mu \neq 0$ in general, so

$$G_{\mu\nu} = c_1 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$$

Condition (4) then determines c_1.

For nonrelativistic systems $|T_{ij}| \ll |T_{00}|$, so $|G_{ij}| \ll |G_{00}|$.

$$\Rightarrow \quad R_{ij} \approx \frac{1}{2} g_{ij} R$$

with $g_{\mu\nu} \approx \eta_{\mu\nu}$,

$$R \approx \sum_{k=1}^{3} R_{kk} - R_{00} \approx \frac{3}{2} R - R_{00}$$

$$\Rightarrow \quad R \approx 2 R_{00}$$

Then $G_{00} = C_1 (R_{00} - \frac{1}{2} g_{00} R)$

$$\approx 2 C_1 R_{00}$$

$$R_{00} = g^{\lambda\nu} R_{\lambda 0 \nu 0} \approx \sum_{k=1}^{3} R_{k0k0} - R_{0000}$$

For a weak field, we may use the linear part of $R_{\lambda\mu\nu\rho}$:

$$R_{\lambda\mu\nu\rho} \approx \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\rho \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\lambda} - \frac{\partial^2 g_{\lambda\rho}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\rho}}{\partial x^\nu \partial x^\lambda}\right]$$

For a static field, $R_{0000} \approx 0$

$$R_{i0j0} \approx \frac{1}{2} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j}$$

$$\Rightarrow \quad G_{00} \approx 2 C_1 \left(\frac{1}{2} \eta^{ij} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j}\right)$$

$$= C_1 \nabla^2 g_{00}.$$

Hence, condition (4) implies $C_1 = 1$.

We have uniquely determined $\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$, known as the Einstein tensor.

The Einstein field equations are,

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu}}$$

Coordinate Conditions

Weinberg 7.4 The Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is symmetric and has 10 independent components. However, the 10 components are related by the 4 Bianchi identities, $G^{\mu}{}_{\nu;\mu} = 0$.

This leaves effectively $10-4=6$ equations for the 10 independent components of $g_{\mu\nu}$.

The remaining 4 degrees of freedom are not fixed by the Einstein equations. This corresponds to the ability to transform any solution by an arbitrary coordinate transformation $x \to x'(x)$.

The nonlinear generalization of the *harmonic gauge* condition $\partial_\mu h^{\mu}{}_{\nu} = \frac{1}{2}\partial_\nu h^{\mu}{}_{\mu}$ in the weak field ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$), is

$$\boxed{g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0}$$ Harmonic coordinate conditions.

To linear order in $h_{\mu\nu}$,

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} \simeq \eta^{\mu\nu} \cdot \frac{1}{2}\eta^{\lambda\kappa}\left(\partial_\mu h_{\kappa\nu} + \partial_\nu h_{\kappa\mu} - \partial_\kappa h_{\mu\nu}\right)$$

$$= \partial_\mu h^{\lambda\mu} - \frac{1}{2}\partial^\lambda h^{\mu}{}_{\mu}$$

$= 0$ in harmonic coordinates.

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The Cauchy Problem

Initial conditions: Suppose $g_{\mu\nu}(t_0, \vec{x})$, $\frac{\partial}{\partial x^0} g_{\mu\nu}\Big|_{t_0,\vec{x}}$ are known.

If we knew $\frac{\partial^2 g_{\mu\nu}}{\partial (x^0)^2}$ from the Einstein equations

then we could integrate to find $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^0}$ at later times.

The spatial components $G_{ij} = -8\pi G_N T_{ij}$ can be used to solve for $\frac{\partial^2 g_{ij}}{\partial (x^0)^2}$.

However, $G_{\mu 0}$ contains no time derivatives higher than $\frac{\partial g_{\mu\nu}}{\partial x^0}$, as can be seen by the Bianchi identities

$$0 = D_\mu G^{\mu\nu} = +\frac{\partial}{\partial x^0} G^{\mu 0} + \frac{\partial}{\partial x^i} G^{\mu i} + \Gamma^\mu_{\nu\lambda} G^{\lambda\nu} + \Gamma^\nu_{\nu\lambda} G^{\mu\lambda}$$

$$\rightarrow \boxed{\frac{\partial}{\partial x^0} G^{\mu 0} \equiv \underbrace{-\frac{\partial}{\partial x^i} G^{\mu i} - \Gamma^\mu_{\nu\lambda} G^{\lambda\nu} - \Gamma^\nu_{\nu\lambda} G^{\mu\lambda}}_{\text{At most 2 time derivatives}}}$$

$\rightarrow G^{\mu 0}$ has at most 1 time derivative, as claimed.

The Einstein equations do not allow the Cauchy problem for $g_{\mu 0}$ to be solved. This is due to the coordinate independence.

The 4 equations $G_{\mu 0} = -8\pi G_N T_{\mu 0}$
must be imposed as constraints on initial data.

At time $x^0 = t_0$, suppose this eqn. is satisfied.
The Bianchi identity at $x^0 = t_0$ gives

$$\frac{\partial}{\partial x^0}\left(G^{\mu 0} + 8\pi G_N T^{\mu 0}\right)\Big|_{x^0 = t_0} = 0$$

This can be integrated to give $G^{\mu 0} = -8\pi G_N T^{\mu 0}$
at time $t_0 + \Delta t$, and then for all times x^0.

Energy-Momentum Tensor of Gravitation

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Consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \to 0$ as $x^\mu \to \infty$, but $h_{\mu\nu}$ is not necessarily small.

The part of $R_{\mu\nu}$ linear in $h_{\mu\nu}$ is

$$R^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right)$$

where indices on $h_{\mu\nu}$, $R^{(1)}_{\mu\nu}$, and $\frac{\partial}{\partial x^\lambda}$ are raised and lowered by $\eta_{\alpha\beta}$, not $g_{\alpha\beta}$. For example, $h^\lambda{}_\lambda \equiv \eta^{\lambda\nu} h_{\nu\lambda}$

True tensors like $R_{\mu\nu}$ will continue to have indices raised and lowered by $g_{\mu\nu}$.

Einstein equations:

$$\boxed{R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)\lambda}{}_\lambda = -8\pi G_N \left(T_{\mu\nu} + t_{\mu\nu}\right)}$$

where $\boxed{t_{\mu\nu} \equiv \frac{1}{8\pi G_N}\left\{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\lambda{}_\lambda - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} R^{(1)\lambda}{}_\lambda\right\}}$

Einstein's equations have the form of a wave-like equation for $h_{\mu\nu}$ with source $T_{\mu\nu} + t_{\mu\nu}$.
IF $T_{\mu\nu}$ is the energy-momentum tensor of matter, it is natural to consider $t_{\mu\nu}$ the energy-momentum "tensor" of gravitation. ($t_{\mu\nu}$ does not transform as a tensor.)

The *total* energy-momentum tensor of matter and gravitation is $\boxed{\tau_{\mu\nu} \equiv T_{\mu\nu} + t_{\mu\nu}}$.

Properties of $\tau_{\mu\nu}$:

1) $\tau_{\mu\nu} = \tau_{\nu\mu}$ $\qquad \tau^{\nu\lambda} \equiv \eta^{\nu\sigma}\eta^{\lambda\rho}\tau_{\sigma\rho}$

2) $\boxed{\frac{\partial}{\partial x^\nu}\tau^{\nu\lambda} = 0}$, i.e. $\tau^{\nu\lambda}$ is *conserved* in the ordinary sense. (by the linearized Bianchi ID, $\partial_\mu(R^{\mu\nu(1)} - \frac{1}{2}\eta^{\mu\nu}R^{(1)}) = 0$)

Hence, $P^\lambda = \int d^3x\, \tau^{0\lambda}$ has the interpretation of the energy-momentum "vector" of the system, including gravitation. — Not generally covariant, but Lorentz covariant.

3) Expanding in $h_{\mu\nu}$, the first term in $t_{\mu\nu}$ is quadratic:

$$t_{\mu\nu} = \frac{1}{8\pi G}\Big[-\frac{1}{2}h_{\mu\nu}R^{(1)\lambda}{}_\lambda + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma} + R^{(2)}_{\mu\nu}$$
$$- \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}\Big] + O(h^3)$$

where

$$R^{(2)}_{\mu\nu} = -\frac{1}{2}h^{\lambda\kappa}\Big[\partial_\nu\partial_\mu h_{\lambda\kappa} - \partial_\nu\partial_\lambda h_{\mu\kappa} - \partial_\kappa\partial_\mu h_{\lambda\nu} + \partial_\kappa\partial_\lambda h_{\mu\nu}\Big]$$
$$+ \frac{1}{4}\Big[2\partial_\kappa h^\kappa{}_\sigma - \partial_\sigma h^\kappa{}_\kappa\Big]\Big[\partial_\nu h^\sigma{}_\mu + \partial_\mu h^\sigma{}_\nu - \partial^\sigma h_{\mu\nu}\Big]$$
$$- \frac{1}{4}\Big[\partial_\lambda h_{\sigma\nu} + \partial_\nu h_{\sigma\lambda} - \partial_\sigma h_{\lambda\nu}\Big]\Big[\partial^\lambda h^\sigma{}_\mu + \partial_\mu h^{\sigma\lambda} - \partial^\sigma h^\lambda{}_\mu\Big]$$