Number of Independent Components in $R^{\mu}{}_{\nu\lambda\sigma}$ (in 4D)

$R_{\mu\nu\lambda\sigma}$ is symmetric in exchange of $(\mu\nu)$ and $(\lambda\sigma)$, and antisymmetric in $\mu\nu$, and antisymmetric in $\lambda\sigma$. There are $\frac{4\cdot 3}{2} = 6$ independent choices for $\mu\nu$, and 6 for $\lambda\sigma$.

Then there are $\frac{6\cdot 7}{2} = 21$ choices for $\mu\nu\lambda\sigma$.

The cyclic sum $R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$, which is really one additional constraint because the cyclic sum is completely antisymmetric in exchange of its indices.

Finally there are $21 - 1 = \boxed{20}$ independent components of the curvature tensor $R^{\mu}{}_{\nu\lambda\sigma}$.

The Ricci tensor $R_{\mu\nu}$ is symmetric in $\mu\nu$, so it has $\frac{4\cdot 5}{2} = \boxed{10}$ independent components.

This implies that the traceless part of $R^{\mu}{}_{\nu\lambda\sigma}$ has $20 - 10 = 10$ independent components. It is called the Weyl tensor, and with all lower indices has the form

$$\boxed{\begin{aligned} C_{\lambda\mu\nu\kappa} \equiv R_{\lambda\mu\nu\kappa} &- \frac{1}{2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) \\ &+ \frac{1}{6}R\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right)\end{aligned}}$$

In 2D, $R_{\lambda\mu\nu\rho}$ has $1\cdot 2/2 = \boxed{1}$ independent component.

choice for ($\lambda\mu$ or $\nu\rho$) — symmetric in ($\lambda\mu$) $\leftrightarrow$ ($\nu\rho$)

In this case $R_{\lambda\mu\nu\rho}$ can be written in terms of the curvature scalar R: $R_{\lambda\mu\nu\rho} = \frac{1}{2}R(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu})$

The Gaussian curvature K is defined for a 2D manifold as $\boxed{K \equiv -R/2}$

The Gaussian curvature is coordinate invariant, so it is a description of the geometry <u>intrinsic</u> to the manifold, independent of any coordinate choices.

In 1D, $R_{\lambda\mu\nu\rho} = 0$ by antisymmetry in $\lambda\mu$ or $\nu\rho$ and the fact that there is only one choice for $\lambda=\mu=\nu=\rho=1$.

Hence all 1D manifolds are flat. The metric can always be chosen to be $g_{11} = \pm 1$ by a coordinate transformation

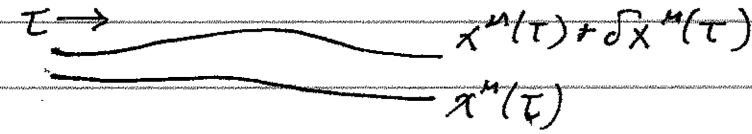
$$g'_{11} = \left(\frac{dx}{dx'}\right)^2 g_{11}.$$

The arc-length along the curve can be used as the coordinate, so that $ds^2 = dx^2$.
This highlights the intrinsic nature of the curvature we have defined. It doesn't matter whether the manifold is, or could be, embedded in a higher-dimensional space.

Geodesic Deviation

Weinberg 6.10

Consider a pair of nearby geodesics in a space(time) with affine connection $\Gamma^{\mu}_{\nu\lambda}$.



Geodesic eqn for each path:

(1) $$0 = \frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda}(x)\,\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}$$

(2) $$0 = \frac{d^2}{d\tau^2}\left[x^{\mu}+\delta x^{\mu}\right] + \Gamma^{\mu}_{\nu\lambda}(x+\delta x)\,\frac{d}{d\tau}\left[x^{\nu}+\delta x^{\nu}\right]\frac{d}{d\tau}\left[x^{\lambda}+\delta x^{\lambda}\right]$$

$$\approx \frac{d^2}{d\tau^2}\left[x^{\mu}+\delta x^{\mu}\right] + \partial_{\rho}\Gamma^{\mu}_{\nu\lambda}(x)\,\delta x^{\rho}\,\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}$$

$$+\Gamma^{\mu}_{\nu\lambda}(x)\,\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} + 2\Gamma^{\mu}_{\nu\lambda}(x)\,\frac{dx^{\nu}}{d\tau}\frac{d\delta x^{\lambda}}{d\tau}$$

(2)−(1): $$\boxed{\frac{d^2}{d\tau^2}\delta x^{\mu} = -\partial_{\rho}\Gamma^{\mu}_{\nu\lambda}\,\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}\,\delta x^{\rho} - 2\Gamma^{\mu}_{\nu\lambda}\,\frac{dx^{\nu}}{d\tau}\frac{d}{d\tau}\delta x^{\lambda}}$$

We can write this covariantly in terms of covariant derivatives along $x^{\mu}(\tau)$.

$$\frac{D}{D\tau}\delta x^{\mu} = \frac{d}{d\tau}\delta x^{\mu} + \Gamma^{\mu}_{\nu\lambda}\,\frac{dx^{\lambda}}{d\tau}\,\delta x^{\nu}$$

$$\frac{D^2}{D\tau^2}\delta x^{\mu} = \frac{d}{d\tau}\left[\frac{d}{d\tau}\delta x^{\mu} + \Gamma^{\mu}_{\nu\lambda}\,\frac{dx^{\lambda}}{d\tau}\,\delta x^{\nu}\right]$$
$$+\Gamma^{\mu}_{\rho\sigma}\,\frac{dx^{\rho}}{d\tau}\left(\frac{d}{d\tau}\delta x^{\sigma} + \Gamma^{\sigma}_{\nu\lambda}\,\frac{dx^{\lambda}}{d\tau}\,\delta x^{\nu}\right)$$

$$\frac{D^2}{D\tau^2}\delta x^\mu = \left(\frac{d^2\delta x^\mu}{d\tau^2} + \partial_\rho \Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu + \Gamma^\mu_{\nu\lambda}\frac{d^2x^\lambda}{d\tau^2}\delta x^\nu + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{d\delta x^\nu}{d\tau}\right) + \left(\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{d}{d\tau}\delta x^\sigma + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu\right)$$

Using the boxed eqn. for $\frac{d^2\delta x^\mu}{d\tau^2}$ and the geodesic eqn for $\frac{d^2x^\lambda}{d\tau^2}$

$$\frac{D^2}{D\tau^2}\delta x^\mu = -\partial_\rho \Gamma^\mu_{\nu\lambda}\delta x^\rho \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \cancel{-2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}} \cancel{+2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}} + \partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\delta x^\nu$$

$$= \delta x^\lambda \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\underbrace{\left(\partial_\rho\Gamma^\mu_{\nu\lambda} - \partial_\lambda\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\mu_{\lambda\sigma}\Gamma^\sigma_{\rho\nu}\right)}_{R^\mu{}_{\nu\lambda\rho}}$$

$$\boxed{\frac{D^2}{D\tau^2}\delta x^\mu = \delta x^\lambda \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} R^\mu{}_{\nu\lambda\rho}}$$

Geodesic Deviation Equation

Consider a gravitational wave with polarization tensor

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & 0 & 0 \\ 0 & 0 & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and metric $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$

with $h_{\mu\nu} = f(z-t)\,\epsilon_{\mu\nu}$.

Consider two nearby particles separated by vector δx^λ,

$\frac{dx^\nu}{d\tau} \approx \delta^\nu{}_0$. (nonrelativistic)

The geodesic deviation equation is in this limit

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu{}_{0\lambda 0}\,\delta x^\lambda$$

The relevant Christoffel symbols are $\Gamma^\mu_{0\lambda} \approx \frac{1}{2}\eta^{\mu\sigma}\partial_0 h_{\lambda\sigma} = \frac{1}{2}\partial_0 h_\lambda{}^\mu$

$$\Gamma^\mu_{00} = 0$$

$$R^\mu{}_{0\lambda 0} = \partial_0 \Gamma^\mu_{0\lambda} - \cancel{\partial_\lambda \Gamma^\mu_{00}} + \underbrace{\Gamma^\mu_{\rho 0}\Gamma^\rho_{0\lambda}}_{\mathcal{O}(h^2)} - \Gamma^\mu_{\rho\lambda}\cancel{\Gamma^\rho_{00}}$$

$$\approx \frac{1}{2}\partial_0^2 h_\lambda{}^\mu = \frac{1}{2}f''(z-t)\,\epsilon_\lambda{}^\mu$$

$$\frac{D^2}{D\tau^2}\delta x^1 = \frac{1}{2}f''(z-t)\,\epsilon_{11}\,\delta x^1$$

$$\frac{D^2}{D\tau^2}\delta x^2 = \frac{1}{2}f''(z-t)\,(-\epsilon_{11})\,\delta x^2$$

For nonrelativistic particles in a weak field, $\tau \approx t$, $\frac{D}{D\tau} \approx \frac{d}{dt}$.

$$\Rightarrow \frac{d^2}{dt^2}\delta x^1 = \frac{1}{2}f''(z-t)\,\epsilon_{11}\,\delta x^1$$

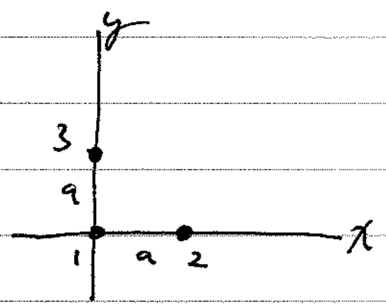
$$\frac{d^2}{dt^2}\delta x^2 = -\frac{1}{2}f''(z-t)\,\epsilon_{11}\,\delta x^2$$

To $\mathcal{O}(\delta x)^2$, the solution is:

$$\boxed{\begin{aligned} \delta x^1(t) &\simeq \left(1+\tfrac{1}{2} f(z-t)\epsilon_{11}\right)\delta x^1(0) \\ \delta x^2(t) &\simeq \left(1-\tfrac{1}{2} f(z-t)\epsilon_{11}\right)\delta x^2(0) \end{aligned}}$$

This should be compared to our earlier analysis of gravitational waves, in which we calculated the proper distance between two nearby particles in the background of a gravitational wave.

From before:



$$\Delta\tau_{12} \approx a\left(1-\lambda h_{xx}\right) = a\left(1-\lambda\epsilon_{xx} f(z-t)\right)$$
$$\Delta\tau_{13} \approx a\left(1-\lambda h_{yy}\right) = a\left(1+\lambda\epsilon_{xx} f(z-t)\right)$$

where $g_{\mu\nu} = \eta_{\mu\nu} - 2\lambda h_{\mu\nu}$

Replacing $\lambda = -\frac{1}{2}$, $a = \delta x^i(0)$, $\epsilon_{xx} = \epsilon_{11}$, we confirm the physical interpretation of $\Delta\tau$ and δx^i as physical displacements.