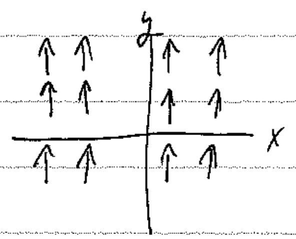
Constant Vector Fields

Consider flat space, with Cartesian coordinates ξ^α,
$ds^2 = d\xi^\alpha d\xi^\beta \delta_{\alpha\beta}$ in ordinary space, or
$ds^2 = d\xi^\alpha d\xi^\beta \eta_{\alpha\beta}$ in spacetime.

A constant vector field is one in which $\frac{\partial V^\mu}{\partial \xi^\alpha} = 0$.

(Note that in non-Cartesian coordinates this is not true,
$\frac{\partial V^\mu}{\partial x^\nu} \neq 0$ in general.)



$V^x = 0$, $V^y = 1$ constant, but
$V^r = \sin\theta$, $V^\theta = \frac{\cos\theta}{r}$ polar coords.

In curved space(time), a constant vector field satisfies $\frac{\partial V^\mu}{\partial \xi^\alpha} = 0$ in locally flat (inertial) coordinates at each point.

General coordinates:

V^δ in x-coords.

locally inertial

$$\frac{\partial V^\mu_{LI}(\xi)}{\partial \xi^\alpha} = \frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial}{\partial x^\nu}\Big[\underbrace{V^\delta_{(x)}\frac{\partial \xi^\mu}{\partial x^\delta}}_{V^\delta_{LI}\text{ in }\xi\text{-coords}}\Big]$$

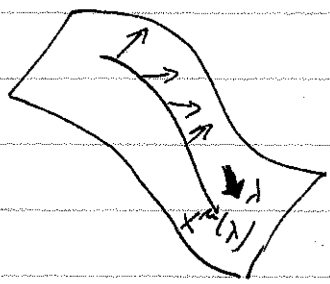
$$= \frac{\partial x^\nu}{\partial \xi^\alpha}\left[\frac{\partial \xi^\mu}{\partial x^\delta}\frac{\partial V^\delta}{\partial x^\nu} + V^\delta \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\delta}\right]$$

$$= \frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial \xi^\mu}{\partial x^\delta}\Big[\frac{\partial V^\delta}{\partial x^\nu} + V^\sigma \underbrace{\frac{\partial x^\delta}{\partial \xi^\beta}\frac{\partial^2 \xi^\beta}{\partial x^\nu \partial x^\sigma}}_{\Gamma^\delta_{\nu\sigma}}\Big]$$

$$\boxed{\frac{\partial V^\mu_{LI}}{\partial \xi^\alpha} = \frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial \xi^\mu}{\partial x^\delta} V^\delta{}_{;\nu}}$$

The condition for a vector field to be constant in curved space is $V^{\mu}{}_{;\sigma} = 0$.

Covariant Derivative Along a Curve



Locally Cartesian (inertial) coordinates $\xi^{\alpha}(\lambda)$

$$\frac{DV^{\mu}_{LI}}{D\lambda} = \lim_{\Delta \to 0} \frac{V^{\mu}_{LI}(\lambda+\Delta) - V^{\mu}_{LI}(\lambda)}{\Delta}$$

$$\frac{DV^{\mu}_{LI}}{D\lambda} = \frac{d\xi^{\alpha}}{d\lambda}\frac{\partial V^{\mu}_{LI}}{\partial \xi^{\alpha}} = \frac{d\xi^{\alpha}}{d\lambda}\frac{\partial x^{\nu}}{\partial \xi^{\alpha}}\frac{\partial \xi^{\mu}}{\partial x^{\sigma}} V^{\sigma}{}_{;\nu}$$

$$= \frac{\partial \xi^{\mu}}{\partial x^{\sigma}} \underbrace{\left(\frac{dx^{\nu}}{d\lambda} V^{\sigma}{}_{;\nu}\right)}_{\frac{DV^{\sigma}}{D\lambda}}$$

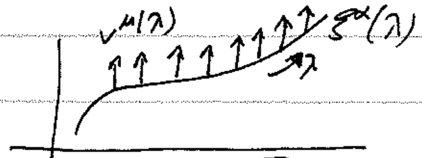
$$\frac{DV^{\sigma}}{D\lambda} = \frac{dx^{\nu}}{d\lambda}\left(\frac{\partial V^{\sigma}}{\partial x^{\nu}} + \Gamma^{\sigma}_{\mu\nu} V^{\mu}\right)$$

$$\boxed{\frac{DV^{\sigma}}{D\lambda} = \frac{dV^{\sigma}}{d\lambda} + \Gamma^{\sigma}_{\mu\nu}\frac{dx^{\nu}}{d\lambda} V^{\mu}}$$

Note that this last expression defines the covariant derivative along a curve even for vector fields defined only along the curve (like $x^{\mu}(\tau)$ describing the trajectory of a particle).

Parallel Transport of Vectors

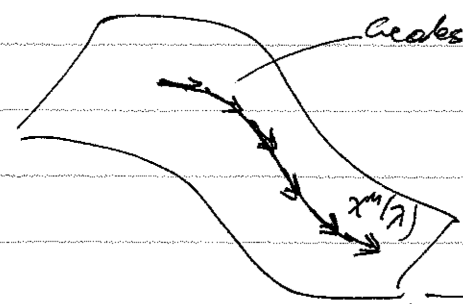
Keep vector constant with respect to itself along a trajectory

Flat space: 

$\frac{DV^\mu}{D\lambda} = 0$ ← Defines parallel transport

$$\boxed{\frac{dV^\mu}{d\lambda} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\lambda}V^\nu}$$ parallel transport equation.

Along a geodesic the tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$ is parallel transported.



$$\frac{D}{D\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0 \longrightarrow \boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0}$$

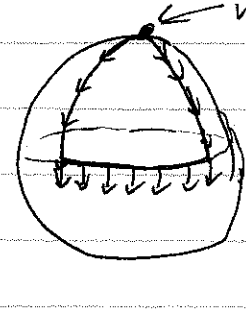
Geodesic Eqn.

Weinberg, Chilo

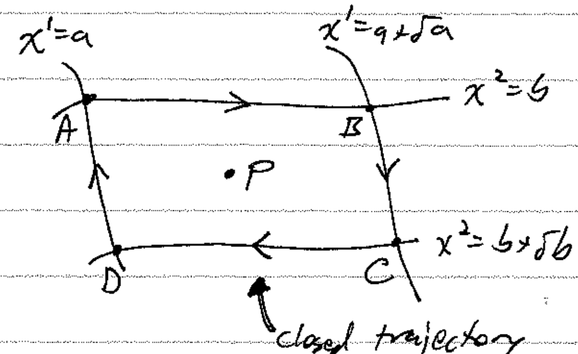
Curvature

In curved spaces, parallel transport of a vector along a closed loop does not generally leave a vector invariant upon traversing a full cycle.

← vector does not return to itself.



<u>Def</u>: A manifold is flat if any vector parallely transported along any closed loop returns the vector to itself.



lines of constant coordinates

Along path from A to B: $\dfrac{DV^\alpha}{D\lambda}=0$

$$\frac{dV^\alpha}{d\lambda} = -\Gamma^\alpha_{\mu 1}\frac{dx^1}{d\lambda}V^\mu \quad , \quad \frac{\partial V^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\mu\nu}V^\nu = 0 \text{ along trajectory.}$$

$$V^\alpha(B) = V^\alpha(A) + \int_{x^2=b} dx^1\left(-\Gamma^\alpha_{\mu 1}V^\mu\right)$$

Similarly $\quad V^\alpha(C) = V^\alpha(B) + \int_{x^1=a+\delta a}\left(-\Gamma^\alpha_{\mu 2}V^\mu\right)dx^2$

$$V^\alpha(D) = V^\alpha(C) + \int_{x^2=b+\delta b} dx^1\left(\Gamma^\alpha_{\mu 1}V^\mu\right)$$

$$V^\alpha(A_{return}) = V^\alpha(D) + \int_{x^1=a}\left(\Gamma^\alpha_{\mu 2}V^\mu\right)dx^2$$

$$V^\alpha(A_{return}) - V^\alpha(A) = \int_{x^1=a} dx^2\,\Gamma^\alpha_{\mu 2}V^\mu - \int_{x^1=a+\delta a}\Gamma^\alpha_{\mu 2}V^\mu\,dx^2$$

$$+ \int_{x^2=b+\delta b} dx^1\,\Gamma^\alpha_{\mu 1}V^\mu - \int_{x^2=b}\Gamma^\alpha_{\mu 1}V^\mu\,dx^1$$

$$= \int_b^{b+\delta b} dx^2\,\delta a\left(-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 2}V^\mu\right)\right) + \int_a^{a+\delta a} dx^1\,\delta b\left(\frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\mu 1}V^\mu\right)\right)$$

$$= \delta a\,\delta b\left[-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 2}V^\mu\right) + \frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\mu 1}V^\mu\right)\right]$$

$V^\alpha(A_{return}) - V^\alpha(A) =$

$$= \delta a\, \delta b \left\{ -\left(\frac{\partial}{\partial x^1}\Gamma^\alpha_{\mu 2}\right)V^\mu - \Gamma^\alpha_{\mu 2}\frac{\partial}{\partial x^1}V^\mu + \left(\frac{\partial}{\partial x^2}\Gamma^\alpha_{\mu 1}\right)V^\mu + \Gamma^\alpha_{\mu 1}\frac{\partial}{\partial x^2}V^\mu \right\}$$

The vector V^μ is parallel transported along the loop, so

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1}V^\mu \quad \text{or} \quad \frac{\partial V^\alpha}{\partial x^2} = -\Gamma^\alpha_{\mu 2}V^\mu$$

along appropriate portions of the loop.

$$\Rightarrow V^\alpha(A_{return}) - V^\alpha(A) \equiv \delta V^\alpha$$

$$= \delta a\, \delta b \left\{ \frac{\partial}{\partial x^2}\Gamma^\alpha_{\mu 1} - \frac{\partial}{\partial x^1}\Gamma^\alpha_{\mu 2} + \Gamma^\alpha_{\beta 2}\Gamma^\beta_{\mu 1} - \Gamma^\alpha_{\beta 1}\Gamma^\beta_{\mu 2} \right\} V^\mu$$

$$\delta V^\alpha \equiv \delta a\, \delta b\, R^\alpha{}_{\mu 1 2} V^\mu$$

2 ← δb in x^2-direction

1 ← δa in x^1-direction

More generally, if V^μ is parallel transported around a loop spanning δa in x^σ-direction, δb in x^λ direction $(\sigma \neq \lambda)$:

$$\boxed{\delta V^\alpha = \delta a\, \delta b\, R^\alpha{}_{\mu\sigma\lambda} V^\mu}$$

$$\boxed{R^\alpha_{\mu\sigma\lambda} \equiv \frac{\partial \Gamma^\alpha_{\mu\sigma}}{\partial x^\lambda} - \frac{\partial \Gamma^\alpha_{\mu\lambda}}{\partial x^\sigma} + \Gamma^\alpha_{\delta\lambda}\Gamma^\delta_{\mu\sigma} - \Gamma^\alpha_{\delta\sigma}\Gamma^\delta_{\lambda\mu}}$$

<u>Riemann Curvature tensor</u>

<u>Exercise</u>: Show that $R^\alpha_{\mu\sigma\lambda}$ is a tensor.

★ Space(time) is flat iff. $R^\alpha_{\mu\sigma\lambda} = 0$ everywhere.

Properties of $R^\alpha{}_{\mu\sigma\lambda}$:

1) $R^\alpha{}_{\mu\sigma\lambda}$ is the only tensor that can be constructed from $g_{\mu\nu}$ and its first and second derivatives.

2) $R^\alpha{}_{\mu\sigma\lambda}$ can also be defined in terms of the commutator of covariant derivatives:

$$\boxed{V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -V_\sigma R^\sigma{}_{\mu\nu\kappa}}$$

$$\boxed{V^\lambda{}_{;\nu;\kappa} - V^\lambda{}_{;\kappa;\nu} = V^\sigma R^\lambda{}_{\sigma\nu\kappa}}$$

3) Define $R_{\lambda\mu\nu\kappa} \equiv g_{\lambda\sigma} R^\sigma{}_{\mu\nu\kappa}$

$$\boxed{\begin{aligned} R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left\{ \frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right\} \\ + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda} \Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda} \Gamma^\sigma_{\mu\nu} \right] \end{aligned}}$$

4)
$$\left.\begin{aligned} &R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \\ &R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu} \\ &R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \end{aligned}\right\} \text{Algebraic Relations}$$

Useful contractions of $R^{\lambda}{}_{\mu\nu\kappa}$:

$$R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa} \qquad \text{Ricci tensor}$$

$$R \equiv g^{\mu\kappa} R_{\mu\kappa} \qquad \text{Curvature Scalar}$$

Bianchi Identities

In a locally Cartesian (inertial) coordinate system, $\Gamma^{\lambda}_{\mu\nu} = 0$, but $\frac{\partial}{\partial x^{\alpha}} \Gamma^{\lambda}_{\mu\nu} \neq 0$.

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2} \frac{\partial}{\partial x^{\eta}} \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa} \partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu} \partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu} \partial x^{\lambda}} \right)$$

Exercise: $\boxed{R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0}$ cyclic permutations

This is a covariant relation so it is true in arbitrary frames.

Contract with $g^{\lambda\nu}$:

$$\boxed{R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu} = 0}$$

Contract w/ $g^{\mu\kappa}$:

$$R_{;\eta} - R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu} = 0$$

$$\Rightarrow \left(R^{\mu}{}_{\eta} - \frac{1}{2}\delta^{\mu}{}_{\eta} R\right)_{;\mu} = 0$$

$$\boxed{\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0}$$