Weinberg 4.4-4.8

Tensor Densities - transform with extra factors of $\det\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)$ or $\det\left(\frac{\partial x'^\rho}{\partial x^\mu}\right)$ under coordinate transformations $x \to x'$.

Example: $g \equiv |\det g_{\mu\nu}|$ (The notation is different than for any other tensor. $T \equiv T^\mu{}_\mu$ is a trace. For the metric $g^\mu{}_\mu = 4$.)

$g_{\mu\nu}$ ← metric

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$$

$$\det g'_{\mu\nu} = \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^2 \det g_{\rho\sigma}$$

$$= \frac{1}{\det\left(\frac{\partial x'}{\partial x}\right)^2} \det(g_{\rho\sigma}) \to \text{tensor density of weight } -2$$

$\det\left(\frac{\partial x'}{\partial x}\right)$ is the Jacobian of the transformation $x \to x'$

The usual volume measure d^4x is a scalar density of weight 1:

$$d^4x' = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| d^4x$$

The measure $\boxed{\sqrt{g}\, d^4x}$ is a scalar. Integrals of scalars against this measure will be coordinate invariant. (This also gives the volume element in an arbitrary coordinate system.)

Example! $$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \mu\nu\lambda\sigma \text{ is an even permutation of } 0123 \\ -1 & \mu\nu\lambda\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

← Levi-Civita tensor (density)

tensor density of weight -1.

The Affine Connection is not a tensor under general coordinate transformations.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

, $\xi^{\alpha}(x)$ is a locally inertial coordinate system.

In the coordinate system x',

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

chain rule

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right)$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right]$$

$$\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

$$\boxed{\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}}$$

Differentiation of a tensor does not generally yield another tensor.

Under the transformation $x \to x'$, $V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$ for vector V^{μ}.

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}}}_{\text{Tensor-like transformation}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}}_{\text{non-tensorlike.}}$$

semicolon, not the letter j

However, the combination $\boxed{V^{\mu}{}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}}$,

called the <u>covariant derivative</u>, is a tensor:

$$V'^{\mu}{}_{;\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}{}_{;\rho}.$$

To construct a covariant derivative of a covariant vector V_μ, it will be helpful to rewrite the transformation of the $\Gamma^{\lambda}_{\mu\nu}$ in a different way.

Use $\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}$

$$\frac{\partial}{\partial x'^{\mu}}: \quad \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} + \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} = 0$$

$$\Rightarrow \boxed{\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}}}$$

It is then straightforward to show that $\boxed{V_{\mu;\nu} \equiv \frac{\partial V_{\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu} V_{\lambda}}$, the covariant derivative of V_μ, transforms like a tensor,

$$\boxed{V'_{\mu;\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} V_{\rho;\sigma}}$$

(Alternative Notation: $V^{\mu}{}_{;\lambda} \equiv D_{\lambda} V^{\mu}$
$V_{\mu;\lambda} \equiv D_{\lambda} V_{\mu}$)

In general, covariant derivatives of tensors involve a sum of terms, each involving one factor of $\Gamma^{\lambda}_{\mu\nu}$, one for each index on the tensor.

<u>Example</u>: $$T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\sigma}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}{}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}{}_{\lambda} - \Gamma^{\kappa}_{\lambda\rho} T^{\mu\sigma}{}_{\kappa}$$

<u>Exercise</u>: Check that $T^{\mu\sigma}{}_{\lambda;\rho}$ is a tensor.

<u>Properties of Covariant Derivatives</u>

1) $(\alpha A^{\mu}{}_{\nu} + \beta B^{\mu}{}_{\nu})_{;\lambda} = \alpha A^{\mu}{}_{\nu;\lambda} + \beta B^{\mu}{}_{\nu;\lambda}$ linearity

2) $(A^{\mu}{}_{\nu} B^{\lambda})_{;\rho} = A^{\mu}{}_{\nu;\rho} B^{\lambda} + A^{\mu}{}_{\nu} B^{\lambda}{}_{;\rho}$ Leibniz rule

3) $T^{\mu\lambda}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\lambda}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\lambda}{}_{\lambda}$ contraction works as naively expected.

<u>Covariant Differentiation of the Metric</u>

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}$$

$$= 0 \quad \text{(using definition of } \Gamma^{\rho}_{\lambda\mu} \text{ in terms of } g_{\mu\nu})$$

We can also show this by considering a locally inertial coordinate system, in which $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = 0$, $\Gamma^{\rho}_{\lambda\mu} = 0$ at some pt P. But $g_{\mu\nu;\lambda}$ is a tensor, so in a general coordinate system, $g_{\mu\nu;\lambda}$ remains zero.

Similarly, $g^{\mu\nu}{}_{;\lambda} = 0$

$\delta^{\mu}{}_{\nu;\lambda} = 0$

* Covariant Differentiation Commutes w/ Raising, Lowering Indices

$$(g^{\mu\nu}V_\nu)_{;\lambda} = \cancel{g^{\mu\nu}{}_{;\lambda}}^{\,0} V_\nu + g^{\mu\nu}V_{\nu;\lambda}$$

$$= g^{\mu\nu}V_{\nu;\lambda}$$

Special Cases of Covariant Differentiation

Covariant Derivative of a Scalar S: $\boxed{S_{;\mu} = \frac{\partial S}{\partial x^\mu}}$

Covariant Curl: Recall $V_{\mu;\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^{\lambda}_{\mu\nu} V_\lambda$

← symmetric in $\mu \leftrightarrow \nu$.

curl: $\boxed{V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}}$ = ordinary curl.

The covariant divergence of a covariant vector can be written in terms of $g = \det(g_{\mu\nu})$ using the following identity:

$$\boxed{\mathrm{Tr}\left\{M^{-1}(x)\frac{\partial}{\partial x^\lambda}M(x)\right\} = \frac{\partial}{\partial x^\lambda}\ln\det M(x)}$$

Proof: If $x^\lambda \to x^\lambda + \delta x^\lambda$, then

$$\delta \ln\det M = \ln\det(M+\delta M) - \ln\det M$$

$$\delta \ln \det M = \ln\left(\frac{\det(M+\delta M)}{\det M}\right)$$

$$= \ln \det\left(M^{-1}(M+\delta M)\right)$$

$$= \ln \det\left(\mathbb{1} + M^{-1}\delta M\right)$$

$$\approx \ln\left(\mathbb{1} + \operatorname{Tr} M^{-1}\delta M\right)$$

$$\approx \operatorname{Tr} M^{-1}\delta M$$

$$\lim_{\delta x^\lambda \to 0} \frac{\delta \ln \det M}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} \ln \det M$$

$$= \operatorname{Tr}\left(M^{-1}\frac{\partial M}{\partial x^\lambda}\right) \quad \blacksquare$$

With $M = g_{\rho\mu}$, $g^{\mu\rho}\frac{\partial}{\partial x^\lambda} g_{\rho\mu} = \frac{\partial}{\partial x^\lambda} \ln g$

$$= \frac{2}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}\sqrt{g}$$

Then $\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\left\{\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\mu\lambda}\right\}$

$$= \frac{1}{2} g^{\mu\rho}\, \partial_\lambda g_{\rho\mu}$$

$$\boxed{\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{g}}\partial_\lambda \sqrt{g}}$$

$$\Longrightarrow \boxed{V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}\, V^\mu}$$

Covariant Laplacian / D'Alembertian

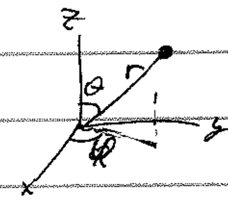
If $\phi(x)$ is a scalar, $\phi_{;\mu}{}^{;\mu} = (g^{\mu\nu}\phi_{;\nu})_{;\mu}$

$$= (g^{\mu\nu}\partial_\nu\phi)_{;\mu}$$

$$\boxed{\phi_{;\mu}{}^{;\mu} = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu\phi\right)}$$

In flat space these formulae allow us to compute the gradient, divergence, and curl in arbitrary coordinates.

Example: Laplacian in spherical coordinates



$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/(r^2\sin^2\theta) \end{pmatrix}$$

$$g \equiv \det g_{\mu\nu} = r^4 \sin^2\theta$$

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\, g^{ij}\partial_j\phi\right)$$

$$= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(r^2\sin\theta\cdot\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\right)\right.$$

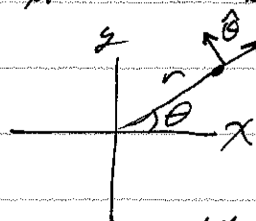
$$\left. + \frac{\partial}{\partial\varphi}\left(r^2\sin\theta\cdot\frac{1}{r^2\sin^2\theta}\frac{\partial\phi}{\partial\varphi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Example: Divergence in 2D Polar coordinates.

$\partial_\mu V^\mu$ is not a scalar. $D_\mu V^\mu \equiv V^\mu{}_{;\mu}$ is a scalar.



Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$ $\Big\|$ $r = \sqrt{x^2+y^2}$, $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

$$V^r = \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y = \cos\theta\, V^x + \sin\theta\, V^y$$
$$V^\theta = \frac{\partial \theta}{\partial x} V^x + \frac{\partial \theta}{\partial y} V^y = -\frac{\sin\theta}{r} V^x + \frac{\cos\theta}{r} V^y$$

$$D_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, V^\mu\right)$$

$$ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow g = r^2$$

$$D_\mu V^\mu = \frac{1}{r}\left[\frac{\partial}{\partial r}(r V^r) + r \frac{\partial}{\partial \theta} V^\theta\right]$$

$$D_\mu V^\mu = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{\partial}{\partial \theta} V^\theta}_{\nabla\cdot\vec{V} \text{ in cylindrical coordinates.}}$$