General Relativity and Cosmology

This is a course about gravitation, namely Einstein's general theory of relativity. There are two complementary approaches to building the theory, both relying on the equivalence principle, which we will discuss during this lecture.

The Field Theoretic Approach:
Builds on our understanding of classical and quantum field theory. Electromagnetism is described by a vector field, which couples to matter in a manner consistent with gauge invariance. Could gravity be similar?

The Geometric Approach:
As we will see, the equivalence principle implies that for a small enough region in space and time, the laws of physics can be written in such a way that they have the same form as in an unaccelerated Cartesian coordinate system w/o gravity as they do in a freely falling frame in the presence of gravity. This statement is similar to the observation by Gauss that on a smooth manifold, for a sufficiently small region one can construct Cartesian coordinates which describe a Euclidean geometry. Could these statements be related?

The geometric description of gravity is the more ingenious of the two approaches, and is the description given by Einstein. We will begin with the field theoretic description because it is based on the successful approach to describing all other interactions, so it is useful to examine the extent to which a similar approach might be successful for gravity, the odd interaction out.

But first we should recall Newton's description of gravitation.

Newtonian Gravity

Newton's theory of gravity, like Einstein's, begins with the observation that all objects seem to fall the same way, independent of their mass.

In the presence of forces $\vec{F}(\vec{x}_N - \vec{x}_M)$ and an external gravitational field $\vec{g}$, Newton's second law takes the form (for particle N):

$$\boxed{m_N \frac{d^2 \vec{x}_N}{dt^2} = m_N \vec{g} + \sum_M \vec{F}(\vec{x}_N - \vec{x}_M)}$$

In a new coordinate system with $\vec{x}' = \vec{x} - \frac{1}{2} g t^2$, $t' = t$, the second law takes the form

$$\boxed{m_N \frac{d^2 \vec{x}_N'}{dt'^2} = \sum_M \vec{F}(\vec{x}_N' - \vec{x}_M')}$$

The coordinate system $(\vec{x}', t')$ describes a freely-falling frame, and in that frame there is no evidence of the gravitational field.

This is a reflection of the equivalence principle, but we can ask to what extent we know it is valid. Suppose there is a difference between the inertial mass m_{Ni} and the gravitational mass m_{Ng}.

Galileo knew that objects appear to fall the same way by comparing the motion of different objects rolling down inclined planes, so crudely we can say $m_{Ni} \approx m_{Ng}$.

Eötvös performed a much more stringent test by comparing the combined forces on hanging plumb bobs due to Earth's gravity (a gravitational effect) and Earth's rotation (an inertial effect). He concluded that for wood (W) and platinum (P),

$$\eta_{W,P} \equiv \frac{2\left(\frac{m_{Wg}}{m_{Wi}} - \frac{m_{Pg}}{m_{Pi}}\right)}{\left(\frac{m_{Wg}}{m_{Wi}} + \frac{m_{Pg}}{m_{Pi}}\right)} \lesssim 10^{-9}$$

With a modern torsion balance experiment, the Eöt-Wash experiment found that the analogous observable for Beryllium and titanium is $\eta_{Be,Ti} \lesssim 10^{-13}$. (PRL 100, 041101 (2008))
No evidence for a difference between inertial and gravitational mass has ever been found.

But this is not the end of the story. What is the gravitational field $\vec{g}$ due to a system of particles?

Gravity, being conservative, can be described in terms of a potential ϕ with $\boxed{\vec{g} = -\nabla\phi}$.

The potential satisfies the Poisson equation
$$\boxed{\nabla^2 \phi = 4\pi G \rho},$$
where G is Newton's constant and $\rho(\vec{x}, t)$ is the mass density.

In anticipation of future discussions, consider instead the equation $\nabla^2 \phi - m^2 \phi = 4\pi G \rho$. (1)
We will analyze the solutions to this equation, and later take $m \to 0$ when we want to describe the gravitational potential.

Fourier transforming $\phi(\vec{x})$, define
$$\tilde{\phi}(\vec{k}) = \int d^3x \; e^{i\vec{k}\cdot\vec{x}} \, \phi(\vec{x}).$$

Multiplying (1) by $e^{i\vec{k}\cdot\vec{x}}$ and integrating over $\vec{x}$,

$$\int d^3x \; e^{i\vec{k}\cdot\vec{x}} \left(\nabla^2 - m^2\right) \phi(\vec{x}) = \int d^3x \; e^{i\vec{k}\cdot\vec{x}} \rho(\vec{x}) \cdot 4\pi G$$

integrate by parts twice
$$= \int d^3x \left(-\vec{k}^2 - m^2\right) e^{i\vec{k}\cdot\vec{x}} \phi(\vec{x}) \quad \equiv \tilde{\rho}(\vec{k}) \cdot 4\pi G$$
$$= -\left(\vec{k}^2 + m^2\right) \tilde{\phi}(\vec{k}).$$

$$\Rightarrow \boxed{\tilde{\phi}(\vec{k}) = -\frac{4\pi G \, \tilde{\rho}(\vec{k})}{\vec{k}^2 + m^2}}$$

The inverse Fourier transform gives $\phi(\vec{x})$:

$$\phi(\vec{x}) = \int \frac{d^3K}{(2\pi)^3} e^{-i\vec{K}\cdot\vec{x}} \tilde{\phi}(\vec{K})$$

$$= -\int \frac{d^3K}{(2\pi)^3} e^{-i\vec{K}\cdot\vec{x}} \frac{4\pi G\tilde{\rho}(\vec{K})}{\vec{K}^2+m^2}$$

$$= -\int \frac{d^3K}{(2\pi)^3} \frac{e^{-i\vec{K}\cdot\vec{x}}}{\vec{K}^2+m^2} \cdot 4\pi G \int d^3x' \, e^{i\vec{K}\cdot\vec{x}'} \rho(\vec{x}') \qquad (2)$$

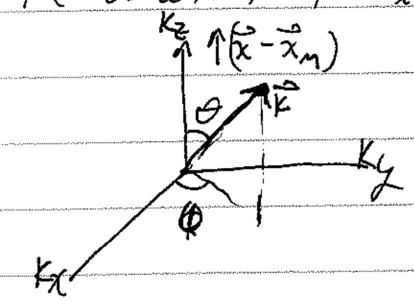
Suppose $\rho(\vec{x}')$ describes a particle of mass M fixed at position $\vec{x}_M$,

$$\rho(\vec{x}') = M\,\delta^3(\vec{x}' - \vec{x}_M)$$

Then doing the integral over $\vec{x}'$ in (2) gives

$$\phi(\vec{x}) = -\frac{4\pi GM}{8\pi^3} \int d^3K \frac{e^{-i\vec{K}\cdot(\vec{x}-\vec{x}_M)}}{\vec{K}^2+m^2}$$

To do the integral over $\vec{K}$, choose the z-axis to point in the direction of $\vec{x}-\vec{x}_M$, and use spherical coordinates



$$\phi(\vec{x}) = -\frac{4\pi GM}{8\pi^3} \int_0^\infty dK \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; K^2 \sin\theta \; \frac{e^{-iK|\vec{x}-\vec{x}_M|\cos\theta}}{K^2+m^2}$$

$$= -\frac{4\pi GM}{8\pi^3} \cdot 2\pi \int_0^\infty dK \frac{K^2}{K^2+m^2} \; \frac{e^{-iK|\vec{x}-\vec{x}_M|} - e^{iK|\vec{x}-\vec{x}_M|}}{-iK|\vec{x}-\vec{x}_M|}$$

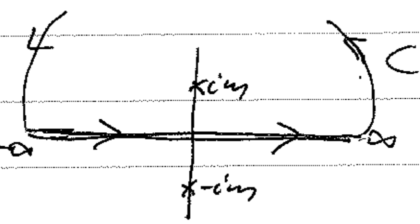
$$= +\frac{GM}{\pi} \frac{1}{|\vec{x}-\vec{x}_M|} \int_{-\infty}^{\infty} dK \; \frac{K^2 e^{+iK|\vec{x}-\vec{x}_M|}}{-iK(K^2+m^2)}$$

To do the final integral over k, analytically continue the integrand to the complex k-plane and use the residue theorem,

$$\oint dk \, \frac{f(k)}{k-k_0} = 2\pi i \, f(k_0)$$

Closing the contour in the upper-half k-plane (appropriate for the exponential $e^{ik|\vec{x}-\vec{x}'|}$), our integral encircles a pole at $k = im$.

$$\phi(\vec{x}) = \frac{GM}{\pi} \frac{1}{|\vec{x}-\vec{x}_M|} \int_C dk \, \frac{k^2 e^{ik|\vec{x}-\vec{x}_M|}}{-ik(k+im)(k-im)}$$

where the contour is 

$$\phi(\vec{x}) = \frac{GM}{\pi} \frac{1}{|\vec{x}-\vec{x}_M|} \cdot \frac{2\pi i \, (im) \exp\left[i(im)|\vec{x}-\vec{x}_M|\right]}{-i\,(2im)}$$

$$\boxed{\phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_M|} e^{-m|\vec{x}-\vec{x}_M|}}$$

With the exponential factor this is known as the Yukawa potential.

With $m \to 0$ this is the usual gravitational potential for a point particle, which is the Coulomb potential (up to an important minus sign):

$$\boxed{\phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_M|}}$$

With spherical coordinates centered at $\vec{x}_M$,
$\phi(\vec{x}) = -\frac{GM}{r}$, and the gravitational field is

$$\vec{g} = -\nabla\phi = -\hat{r}\frac{\partial\phi}{\partial r} = -\frac{GM}{r^2}\hat{r}$$

The gravitational force on a particle of mass m due to the particle of mass M satisfies the $\frac{1}{r^2}$ force law,

$$\boxed{\vec{F}_m = m\vec{g} = -\frac{GMm}{r^2}\hat{r}}$$

The $\frac{1}{r^2}$ force law was suggested by Edmund Halley to Newton as the explanation for Kepler's third law for the period T of planetary orbits as a function of semimajor axis s, $T^2 \propto s^3$. Halley together with Robert Hooke and Christopher Wren had shown this for circular orbits, and Newton proved it for elliptical orbits. Newton had earlier proposed the $\frac{1}{r^2}$ force law by comparing the orbital period of the moon to the gravitational acceleration at Earth's surface.

If we accept the existence of dark matter, no violation of the $\frac{1}{r^2}$ force law has been observed at cosmological distance scales, (galactic scales) planetary scales, tabletop scales, all the way down to ~ 20 nm (Eöt-Wash expt).

The Meaning of Inertia

The coordinate systems in which Newton's laws were presumed to hold were called inertial frames.

Consider the gravitational interaction of a system of point particles,

$$m_N \frac{d^2 \vec{x}_N}{dt^2} = -G \sum_M \frac{m_M m_N (\vec{x}_N - \vec{x}_M)}{|\vec{x}_N - \vec{x}_M|^3} .$$

Consider a new coordinate system $\vec{x}' = R\vec{x} + \vec{v}t + \vec{c}$
$t' = t + t_0$

where R is a rotation, and $\vec{v}$, $\vec{c}$ and t_0 are constants.
3 angles + 3 components + 3 + 1 = 10 parameters

This is the 10-parameter family of Galilean transformations

In the new coordinate system,

$$m_N \frac{d^2}{dt'^2}\left(R^{-1}\vec{x}'\right) = -G \sum_M \frac{m_M m_N R^{-1}(\vec{x}_N' - \vec{x}_M')}{|\vec{x}_N' - \vec{x}_M'|^3}$$

where we used the invariance of length $|\vec{x}_N - \vec{x}_M|$ under rotations

Acting with R on both sides,

$$m_N \frac{d^2 \vec{x}'}{dt'^2} = -G \sum_M \frac{m_M m_N (\vec{x}_N' - \vec{x}_M')}{|\vec{x}_N' - \vec{x}_M'|} .$$

Newton's second law takes the same form in any frame related to an inertial frame by a Galilean transformation.
Invariance of laws of motion under these transformations = Galilean Relativity.

There is a continuing debate as to the interpretation of the inertial frames. Newton himself thought that their existence implied the existence of an absolute notion of space (and time). Ernst Mach suggested that the distribution of matter (eg. the location of the fixed stars) determined the inertial frames.

To understand Mach's perspective consider this experiment: On a clear night rest your arms at your side and gaze at the stars. They appear fixed. Now spin yourself and notice that simultaneously the stars appear to rotate and your arms are lifted by the centrifugal effect. Why should there be a coincidence between the frames in which the stars are fixed (or moving at uniform velocity) and the frames in which there is no centrifugal effect? It appeared to Mach that the heavens determined the inertial frames.

Einstein's view is similar to Mach's, but with important differences as we will see. In particular, the inertial frames will depend on the local distribution of matter and not only on some averaged effect of all the matter in the universe.

Confronting Maxwell

Maxwell's equations for electrodynamics are not consistent with Galilean relativity. In vacuum, from Maxwell's equations follow the wave equation for each component of $\vec{E}$ and $\vec{B}$:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\vec{E}=0$$

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\vec{B}=0$$

where $c=\frac{1}{\sqrt{\mu_0\epsilon_0}}$.

The wavelike solutions all propagate at speed c.

Invariance under the Galilean boost $\vec{x}'=\vec{x}+\vec{v}t$ would imply that a velocity in one frame is shifted in another,

$$\frac{d\vec{x}'}{dt}=\frac{d\vec{x}}{dt}+\vec{v}.$$

This does not leave room for invariant speeds like c.

However, Maxwell's equations are invariant under a different 10-parameter family of transformations known as Poincaré transformations. The statement that physical law should be invariant under Poincaré transformations is the basis of the special theory of relativity. The description of the Poincaré transformations will be our next topic.