Tensors and Tensor Fields

Einstein's equations for the dynamics of the gravitational field are relations between fields which are tensors under general coordinate transformations. We will build up to an understanding of these objects, beginning with a discussion of rotations and tensors under rotations.

We may define rotations as the group of linear transformations of coordinates with one point fixed such that length elements $ds^2 = d\vec{x}^2 = \sum_{i,j=1}^{3} dx^i \delta_{ij} dx^j$ are invariant.

"Kronecker delta" $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

We describe the rotation by a matrix R, such that

$$x^i \xrightarrow{R} x'^i = \sum_{j=1}^{3} R^i{}_j x^j .$$

Then

$$ds^2 \xrightarrow{R} \sum_{i,j,k,l=1}^{3} (R^i{}_k dx^k) \delta_{ij} (R^j{}_l dx^l)$$

$$= \sum_{ijkl} dx^k \underbrace{\left[(R^T)_k{}^i \delta_{ij} R^j{}_l \right]}_{[R^T \mathbb{1} R]_{kl}} dx^l$$

Length elements ds^2 are invariant if $\boxed{R^T \mathbb{1} R = \mathbb{1}}$

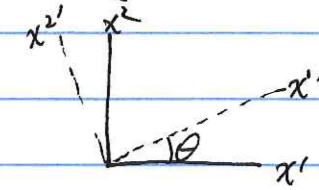
Rotations also satisfy $\boxed{\det R = +1}$. (Reflections satisfy $\det R = -1$)

$\det R = +1$ $\quad$ $R^T R = \mathbb{1}$

Such matrices are called special orthogonal matrices, and form a representation of the rotation group $SO(3)$.

Most of this discussion is easily generalized to dimensions of space other than three. For example, in 2D rotations still satisfy $R^T R = \mathbb{1}$, $\det R = 1$. These matrices, which form a representation of the rotation group $SO(2)$, can be parametrized by an angle θ:

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \to R \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \equiv \begin{pmatrix} x^{1\prime} \\ x^{2\prime} \end{pmatrix}$$

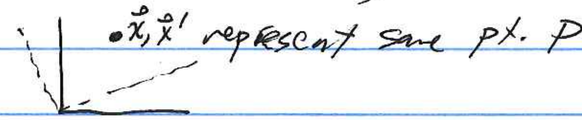


When considering how physical quantities transform under rotations, we will take the passive viewpoint, in which the coordinates change but the physical objects are not themselves rotated.

A scalar field under rotations, $\phi(\vec{x})$, is a field which is invariant except for the relabeling of coordinates.

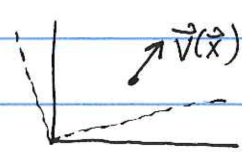
$$\phi(\vec{x}) \xrightarrow{R} \boxed{\phi'(\vec{x}\,') = \phi(\vec{x})}$$

$$\vec{x}\,' = R\vec{x}$$



A vector field transforms like the coordinates do componentwise.

$$\vec{V}(\vec{x}) \xrightarrow{R} \vec{V}'(\vec{x}\,') = R\,\vec{V}(\vec{x})$$



$$\vec{x}\,' = R\vec{x}$$

Componentwise, $\boxed{V'^{\,i}(\vec{x}\,') = \sum_{j=1}^{3} R^i{}_j\, V^j(\vec{x})}$ (back to 3D)

A <u>rank-n tensor field</u> transforms with n factors of the rotation matrix:

$$T^{i_1 \cdots i_n}(\vec{x}) \xrightarrow{R} \boxed{T'^{\,i_1 \cdots i_n}(\vec{x}\,') = \sum_{j_1 \cdots j_n} R^{i_1}{}_{j_1} R^{i_2}{}_{j_2} \cdots R^{i_n}{}_{j_n} T^{j_1 \cdots j_n}(\vec{x})}$$

An equation is said to be <u>rotationally covariant</u> if it transforms consistently under rotations.

$V^i(\vec{x}) = \phi(\vec{x})$ is <u>not</u> covariant: vector transforms while scalar does not.

$\vec{V}(\vec{x}) \cdot \vec{V}(\vec{x}) = \phi(\vec{x})$ is covariant: Both sides are rotational scalars.

In terms of components of $\vec{V}(\vec{x})$, we can write

$$\vec{V} \cdot \vec{V} = \sum_i V^i V^i = \sum_{ij} V^i \delta_{ij} V^j$$

$$\xrightarrow{R} \sum_{ijkl} \left(R^i{}_l V^l\right) \delta_{ij} \left(R^j{}_k V^k\right)$$

$$= \sum_{ijkl} V^l \underbrace{\left[(R^T)_l{}^i \, \mathbb{1}_{ij} \, R^j{}_k\right]}_{(R^T \mathbb{1} R)_{lk} = \mathbb{1}_{lk} = \delta_{lk}} V^k$$

$$= \sum_{kl} V^l \delta_{lk} V^k = \sum_l V^l V^l = \vec{V} \cdot \vec{V}$$

☆ The lesson is that summing over pairs of repeated indices leaves a tensor that transforms as though those indices were not there.

With this understanding, an easy way to test covariance is by comparing indices throughout an equation.

$\sum_j T^{ij}{}_j{}^k = A^{ik}$ ← covariant equation.

Now on to Lorentz transformations = Group of linear transformations of spacetime coordinates which leave the proper length $ds^2 = -c^2 dt^2 + d\vec{x}^2$ invariant.

We define a 4-vector x^μ with components $(\underset{x^0}{ct}, \underset{x^i,\ i=1,2,3}{\vec{x}})$

The proper length can be written

$$ds^2 = \sum_{\mu,\nu=0}^{3} dx^\mu \eta_{\mu\nu} dx^\nu$$

where $\eta_{\mu\nu}$ is the Minkowski tensor $\boxed{\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}_{\mu\nu}}$.

For now we think of $\eta_{\mu\nu}$ as a matrix that does not transform under rotations. It's just a convenient way to write ds^2.

We define $\boxed{x_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\nu}$ with components $(-ct, +\vec{x})$.

The proper length can be written $ds^2 = \sum_{\mu=0}^{3} dx^\mu dx_\mu$.

We also define $\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}^{\mu\nu}$ and write

$$\boxed{x^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} x_\nu}$$

In this way the Minkowski tensor is used to raise and lower indices.

For consistency, $x^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} x_\nu = \sum_{\nu,\alpha=0}^{3} \eta^{\mu\nu} \eta_{\nu\alpha} x^\alpha$,

so $\boxed{\sum_{\nu=0}^{3} \eta^{\mu\nu}\eta_{\nu\alpha} = \delta^\mu{}_\alpha}$ as can be checked explicitly.

$\delta^\mu{}_\alpha = \begin{cases} 1 & \text{if } \mu=\alpha \\ 0 & \text{if } \mu\neq\alpha \end{cases}$

Under Lorentz transformations, 4-vectors transform as

$\boxed{x^\mu \to \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu}$ — contravariant 4-vector (has an upper index.)

Then $x_\mu = \sum_\nu \eta_{\mu\nu} x^\nu \to \sum_{\nu\alpha} \eta_{\mu\nu} \Lambda^\nu{}_\alpha x^\alpha$

$$= \sum_{\nu,\alpha,\beta} \underbrace{\eta_{\mu\nu} \Lambda^\nu{}_\alpha \eta^{\alpha\beta}} x_\beta$$

sometimes written $\Lambda_\mu{}^\beta$ after summing over ν, α.

So, x_μ transforms differently than x^μ.
x_μ is called a covariant 4-vector
(has a lower index)

Lorentz transformation of ds^2:

$$ds^2 = \sum_{\mu\nu} dx^\mu \eta_{\mu\nu} dx^\nu \to \sum_{\mu\nu\alpha\beta} (\Lambda^\mu{}_\alpha dx^\alpha)\eta_{\mu\nu}(\Lambda^\nu{}_\beta dx^\beta)$$

$$= \sum_{\mu\nu\alpha\beta} dx^\alpha (\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta dx^\beta$$

$$= \sum_{\alpha\beta} dx^\alpha \eta_{\alpha\beta} dx^\beta \quad \text{if} \quad \boxed{\sum_{\mu\nu} (\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$$

The group of 4x4 matrices Λ satisfying $\Lambda^T \eta \Lambda = \eta$ are a matrix representation of the Lorentz group $O(3,1)$. The numbers $(3,1)$ refer to the number of $+1$'s and -1's in η, respectively. The restriction $\det \Lambda = +1 \rightsquigarrow SO(3,1)$.

Note that $\sum_{\mu\nu\alpha} \left(\eta_{\mu\nu} \Lambda^\nu{}_\alpha \, \eta^{\alpha\beta} \right) \Lambda^\mu{}_\lambda$

$$= \sum_{\mu\nu\alpha} \left(\Lambda^\mu{}_\lambda \, \eta_{\mu\nu} \Lambda^\nu{}_\alpha \right) \eta^{\alpha\beta}$$

$$= \sum_\alpha \eta_{\lambda\alpha} \eta^{\alpha\beta} = \delta_\lambda{}^\beta .$$

In other words, $\boxed{\sum_{\nu\alpha} \eta_{\mu\nu} \Lambda^\nu{}_\alpha \, \eta^{\alpha\beta} = (\Lambda^{-1})^\beta{}_\mu}$

So we can write the transformation of a covariant 4-vector as

$$\boxed{x_\mu \to \sum_\beta (\Lambda^{-1})^\beta{}_\mu \, x_\beta}$$

A Lorentz tensor can have upper and lower indices, and transforms as

$$T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} \to \sum_{\{\alpha_i, \beta_i\}} \Lambda^{\mu_1}{}_{\alpha_1} \cdots \Lambda^{\mu_n}{}_{\alpha_n} (\Lambda^{-1})^{\beta_1}{}_{\nu_1} \cdots (\Lambda^{-1})^{\beta_m}{}_{\nu_m} T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_m}$$

— Defines an (n,m) tensor.

<u>Exercise</u>: Show that $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are tensors, but as such are invariant under Lorentz transformations.

We now classify fields by their Lorentz transformation properties.

<u>Scalar field</u>: $\phi(x) \xrightarrow{\Lambda} \phi'(x') = \phi(x)$

<u>Vector field</u>: $A^\mu(x) \xrightarrow{\Lambda} A'^\mu(x') = \sum_\nu \Lambda^\mu{}_\nu A^\nu(x)$

$$A_\mu(x) \xrightarrow{\Lambda} \sum_\alpha (\Lambda^{-1})^\alpha{}_\mu A_\alpha(x) \equiv A'_\alpha(x')$$

<u>Tensor field</u>: $T^{\mu_1 \cdots \mu_n}(x) \xrightarrow{\Lambda} T'^{\mu_1 \cdots \mu_n}(x') = \sum_{\alpha_1 \cdots \alpha_n} \Lambda^{\mu_1}{}_{\alpha_1} \cdots \Lambda^{\mu_n}{}_{\alpha_n} T^{\alpha_1 \cdots \alpha_n}(x)$

etc.

A product of 4-vectors $\sum_\mu A_\mu B^\mu$ is Lorentz invariant:

$$\sum_\mu A_\mu B^\mu \longrightarrow \sum_{\alpha\beta\mu} \left((\Lambda^{-1})^\alpha{}_\mu A_\alpha \right) \left(\Lambda^\mu{}_\beta B^\beta \right)$$

$$= \sum_{\alpha\beta\mu} A_\alpha (\Lambda^{-1})^\alpha{}_\mu \Lambda^\mu{}_\beta B^\beta$$

$$= \sum_{\alpha\beta} A_\alpha \delta^\alpha{}_\beta B^\beta = \sum_\alpha A_\alpha B^\alpha \qquad \blacksquare$$

More generally <u>contracting</u> an upper index with a lower index by summing this way, a tensor transforms as though those indices were not there.

To simplify notation, we introduce the Einstein summation convention:

$$A_\mu B^\mu \equiv \sum_{\mu=0}^{3} A_\mu B^\mu.$$

$$T^{\alpha\beta\gamma} T_{\beta\gamma\delta} \equiv \sum_{\beta\gamma=0}^{3} T^{\alpha\beta\gamma} T_{\beta\gamma\delta}$$

Transformation of derivatives:

$$\frac{\partial}{\partial x^\mu}\phi(x) \xrightarrow{\Lambda} \frac{\partial}{\partial x'^\mu}\phi'(x')$$

$$= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\phi(x)$$

with $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $x^\nu = (\Lambda^{-1})^\nu{}_\mu x'^\mu$

$$\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu$$

$$\boxed{\frac{\partial}{\partial x^\mu}\phi(x) \xrightarrow{\Lambda} (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}\phi(x)}$$

This is just how a <u>covariant</u> 4-vector transforms, i.e. $\frac{\partial}{\partial x^\mu}$ can be thought of as transforming like a 4-vector with a <u>lower</u> index.

This motivates some new notation: $\boxed{\partial_\mu \equiv \frac{\partial}{\partial x^\mu}}$

$$\boxed{\partial^\mu \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\nu}}$$

So, $(\partial_\mu\phi)(\partial^\mu\phi) = \frac{\partial\phi}{\partial x^\mu}\eta^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}$

- transforms as a Lorentz scalar.

Still more notation: $(\partial_\mu\phi)^2 \equiv (\partial_\mu\phi)(\partial^\mu\phi)$

- It is the only combination of two factors of $\partial_\mu\phi$ that transforms nicely under Lorentz transformations.

Physical interpretation of Lorentz transformations:

Consider rotations: For a rotation about the x^3-axis by an angle θ we would write

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & \\ & \cos\theta & +\sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

For small angles this becomes, $\begin{pmatrix} 1 & & & \\ & 1 & +\theta & \\ & -\theta & 1 & \\ & & & 1 \end{pmatrix} + \mathcal{O}(\theta^2)$.

We can write the infinitessimal transformation matrix as $\delta^\mu{}_\nu + \omega^\mu{}_\nu$, where $\omega^\mu{}_\nu$ is the antisymmetric matrix $\begin{pmatrix} 0 & & & \\ & 0 & +\theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}$.

For a general infinitessimal rotation by θ about the $\hat{\theta}$ axis we would have for the spatial components $\omega^i{}_j$,
$\omega^{ij} = +\omega^i{}_j = -\omega^j{}_i = -\omega^{ji} = \sum_k \varepsilon^{ijk}\theta\hat{\theta}^k$.

For our rotation about x^3 we have $\omega^{12} = \theta$.

For a boost in the x^1-direction by velocity v,

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \to \begin{pmatrix} \cosh w & \sinh w & & \\ \sinh w & \cosh w & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where the boost parameter w satisfies

$$\cosh w = 1 / \sqrt{1 - v^2/c^2} \equiv \gamma$$

$$\tanh w = v/c$$

For small $\frac{v}{c}$, the transformation matrix becomes,

$$\begin{pmatrix} 1 & v/c & & \\ v/c & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \mathcal{O}(v/c)^2$$

If we again write this as $\delta^\mu{}_\nu + w^\mu{}_\nu$, then in this example $w^0{}_1 = w^1{}_0 = -w^{10} = +w^{01} = v/c$.

A general infinitessimal Lorentz transformation is specified by the <u>antisymmetric matrix $w^{\mu\nu}$</u>.

Count # parameters in $w^{\mu\nu}$: $\frac{4 \cdot 3}{2} = 6$

$= 3$ rotations $+ 3$ boosts ✓

We can understand the antisymmetry of $\omega_{\mu\nu}$ from the defining relation for Lorentz transformations:

$$\Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \omega^\nu{}_\mu \quad , \quad \Lambda^\mu{}_\nu \, \eta_{\mu\beta} \, \Lambda^\beta{}_\alpha = \eta_{\nu\alpha}$$

$$(\delta^\mu{}_\nu + \omega^\mu{}_\nu)\, \eta_{\mu\beta} \, (\delta^\beta{}_\alpha + \omega^\beta{}_\alpha)$$

$$= \eta_{\nu\alpha} + \omega_{\alpha\nu} + \omega_{\nu\alpha} + \underbrace{\omega_{\beta\nu}\omega^\beta{}_\alpha}_{\mathcal{O}(\omega^2)}$$

(Note that we have been raising and lowering indices with $\eta_{\mu\beta}$.)

So, to linear order in ω, $\boxed{\omega_{\alpha\nu} = -\omega_{\nu\alpha}}$, as promised.

Including translations, $x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{6 \text{ parameters}} x^\nu + \underbrace{a^\mu}_{4 \text{ parameters}}$

→ 10-parameter family of <u>Poincaré transformations</u>

Note that there are the same number of parameters in the Poincaré transformations as in the Galilean transformations.

All of the features of special relativity, like Lorentz contraction and time dilation, can be understood in terms of these transformations.

For example:

Consider the rest frame of a clock, with time interval between clicks $-(\Delta s)^2 = (\Delta t)^2$. (Take $c=1$ here)

In a boosted frame, moving with speed v in the x-direction with respect to the clock's frame,

$$-(\Delta s')^2 = (\Delta t')^2 - (\Delta x')^2$$
$$= (\Delta t')^2 - (v\Delta t')^2$$
$$= (\Delta t')^2 (1 - v^2)$$
$$= -(\Delta s)^2$$ because the proper time is invariant.

$$\Rightarrow \boxed{\Delta t' = \frac{1}{\sqrt{1-v^2}} \Delta t}$$ <u>Time dilation</u>

$\left(\text{Compare with } \cosh w = \frac{1}{\sqrt{1-v^2}} \text{ in } \Lambda^{\mu}{}_{\nu}. \right)$